Filtering in frequency domain

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Review: Signals and systems

Reference: Oppenheim & Willsky, “Signals & Systems”, 2nd Ed. 1997, Prentice-Hall

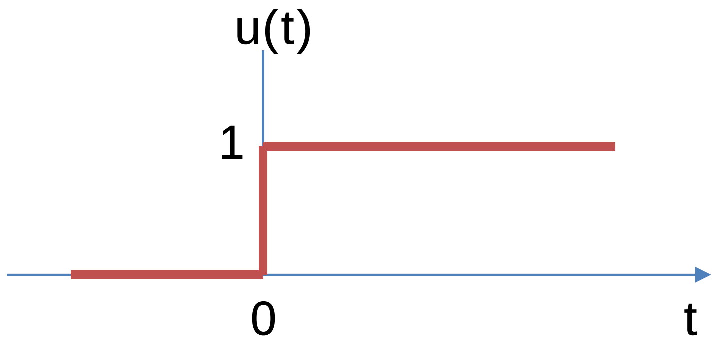
Review

- Step and impulse functions
- Convolution
- Fourier transform
  - Sinc function
  - Duality
- Sampling

Impulse & Step (continuous)

Unit Step Function

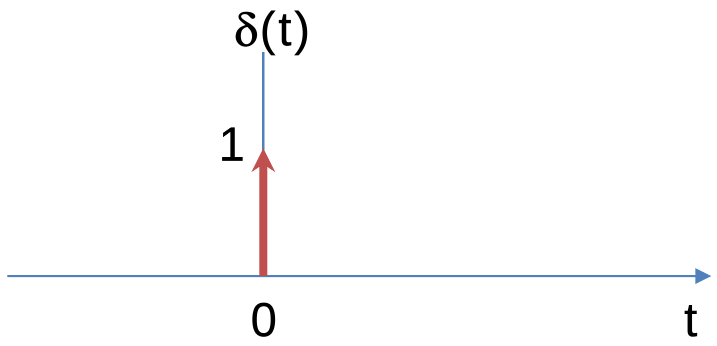
$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$$



$$\delta(t) = \frac{du(t)}{dt} \qquad u(t) = \int_{-\infty}^{t} \delta(\tau) d\tau$$

Unit Impulse Function

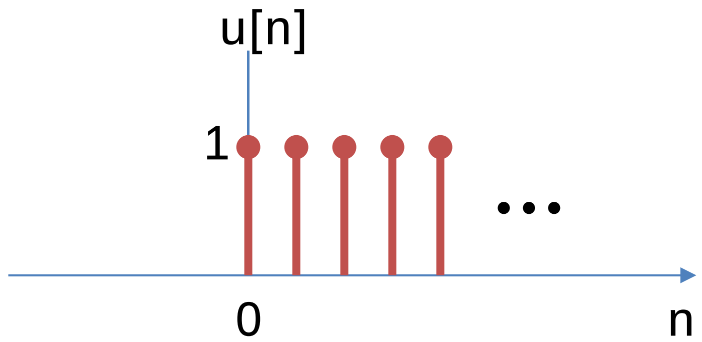
$$\delta(t) = \begin{cases} 1, & t = 0 \\ 0, & otherwise \end{cases}$$



Impulse & Step (discrete)

Unit Step Function

$$u[n] = \begin{cases} 0, & n < 0 \\ 1, & n \geq 0 \end{cases}$$

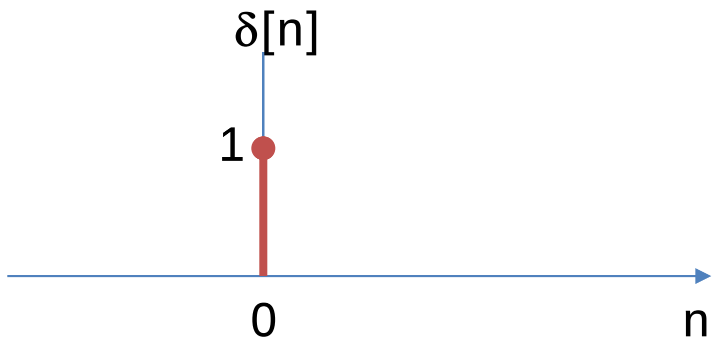


$$\delta[n] = u[n] - u[n-1]$$

$$u[n] = \sum_{k=0}^{\infty} \delta[n-k]$$

Unit Impulse Function

$$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$



Convolution of continuous signals

- Convolution integral

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$$

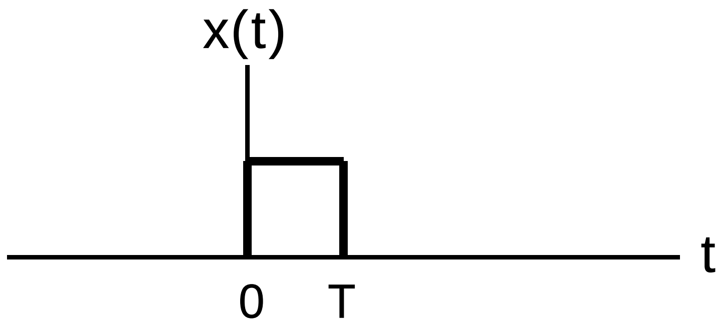
$$y(t) = x(t) * h(t)$$

Question 1: $x(t)*\ \delta(t) = ?$

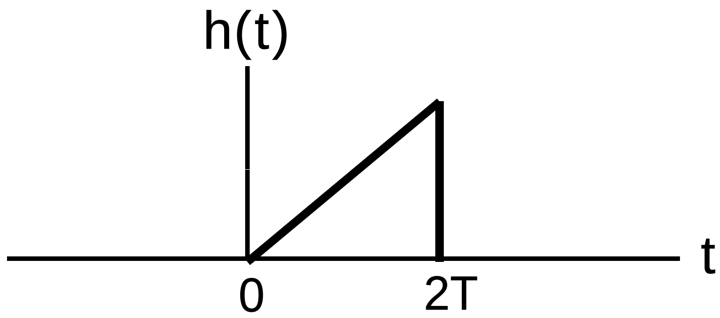
Question 2: $x(t)*\ \delta(t\text{-}t_0) = ?$

Convolution Integral

$$x(t) = \begin{cases} 1, & 0 < t < T \\ 0, & otherwise \end{cases}$$



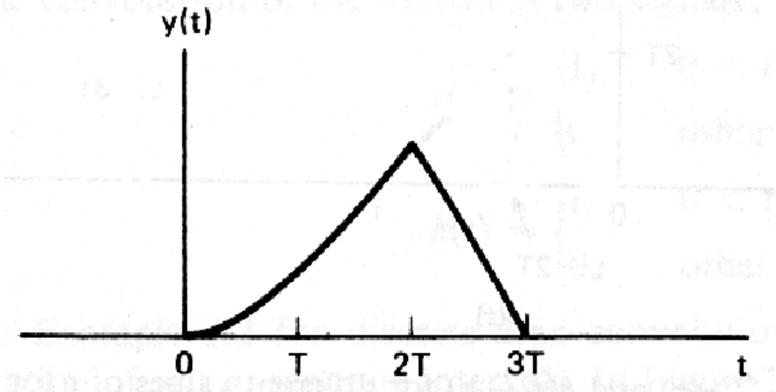
$$h(t) = \begin{cases} t, & 0 < t < 2T \\ 0, & otherwise \end{cases}$$

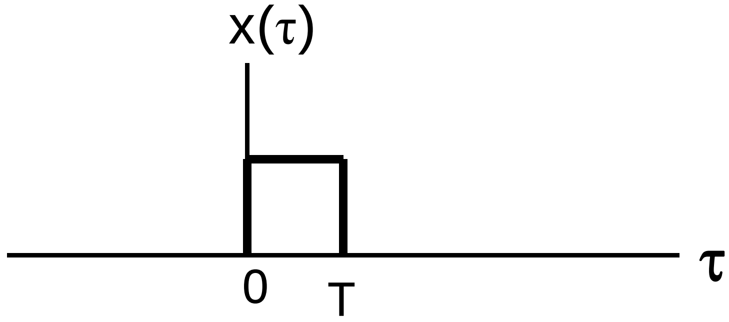


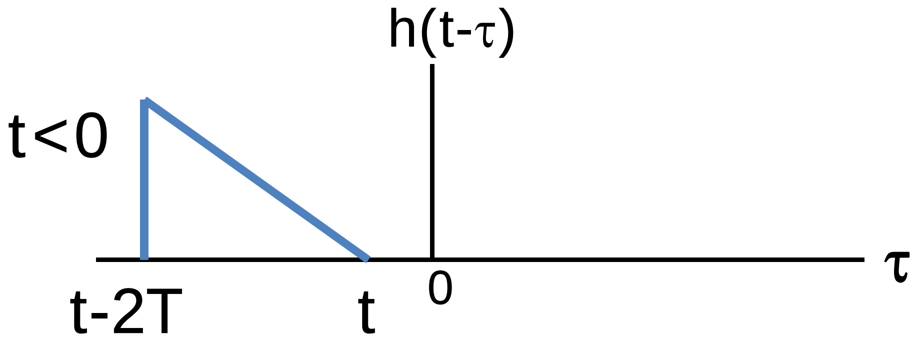
$$y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$$

$$y(t)=\int_{-\infty}^{+\infty}x(\tau)h(t-\tau)d\tau$$

$$y(t)=\begin{cases}0, & t<0\\ \frac{1}{2}t^2, & 0<t<T\\ Tt-\frac{1}{2}T^2, & T<t<2T\\ -\frac{1}{2}t^2+Tt+\frac{3}{2}T^2, & 2T<t<3T\\ 0, & 3T<t\end{cases}$$







好累，還是交給電腦好了…

Convolution of discrete-time signals

- Convolution sum

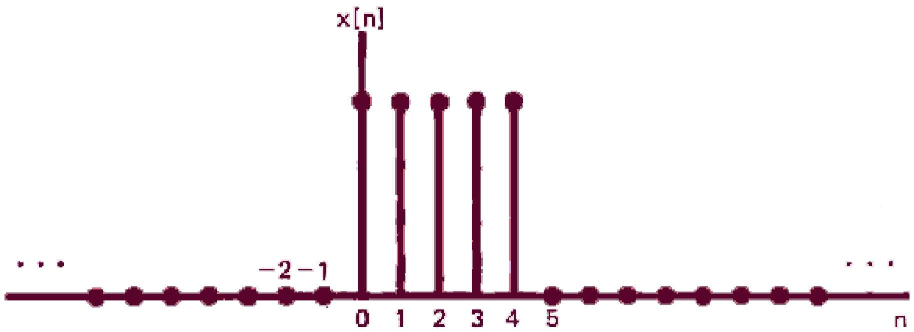
$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$

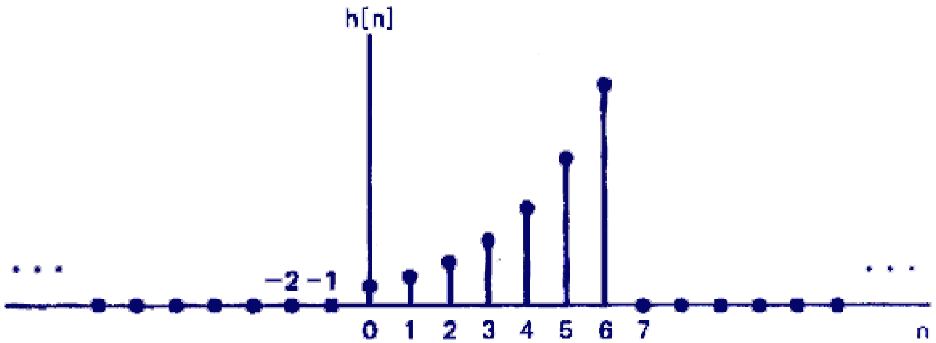
$$y[n] = x[n] * h[n]$$

Convolution Sum

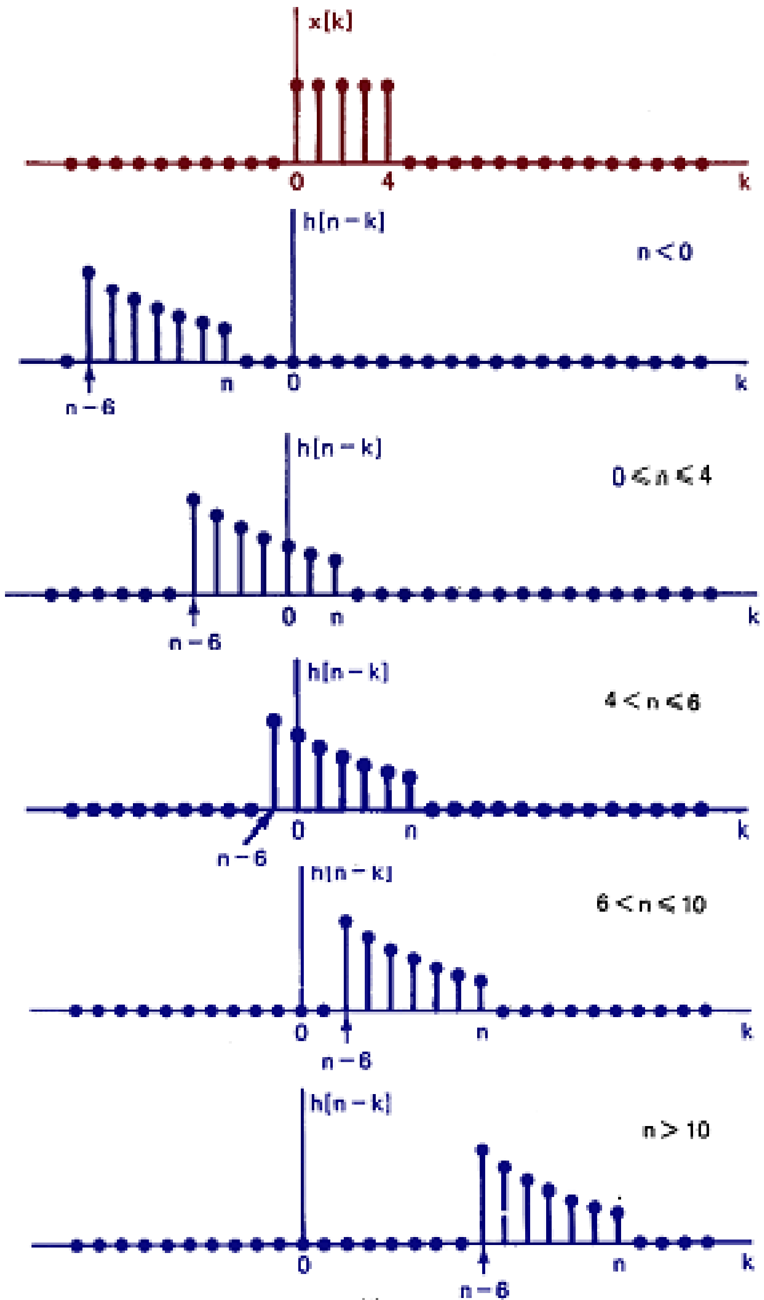
$$x[n]=\begin{cases}1, & 0\le n\le 4\\ 0, & otherwise\end{cases}$$

$$h[n]=\begin{cases}\alpha^n, & 0\le n\le 6\\ 0, & otherwise\end{cases}$$

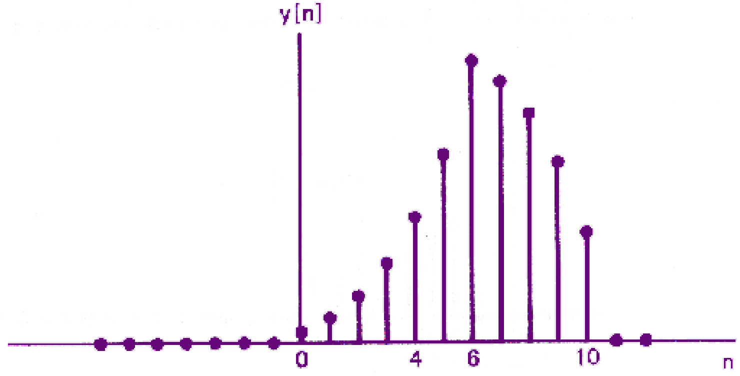




$$y[n]=\sum_{k=-\infty}^{+\infty}x[k]h[n-k]$$



(f)



$$y[n]=\begin{cases}0, \quad n<0 \\ \dfrac{1-\alpha^{n+1}}{1-\alpha}, \quad 0\le n\le 4 \\ \dfrac{\alpha^{n-4}-\alpha^{n+1}}{1-\alpha}, \quad 4<n\le 6 \\ \dfrac{\alpha^{n-4}-\alpha^{7}}{1-\alpha}, \quad 6<n\le 10 \\ 0, \quad 10<n\end{cases}$$

Properties

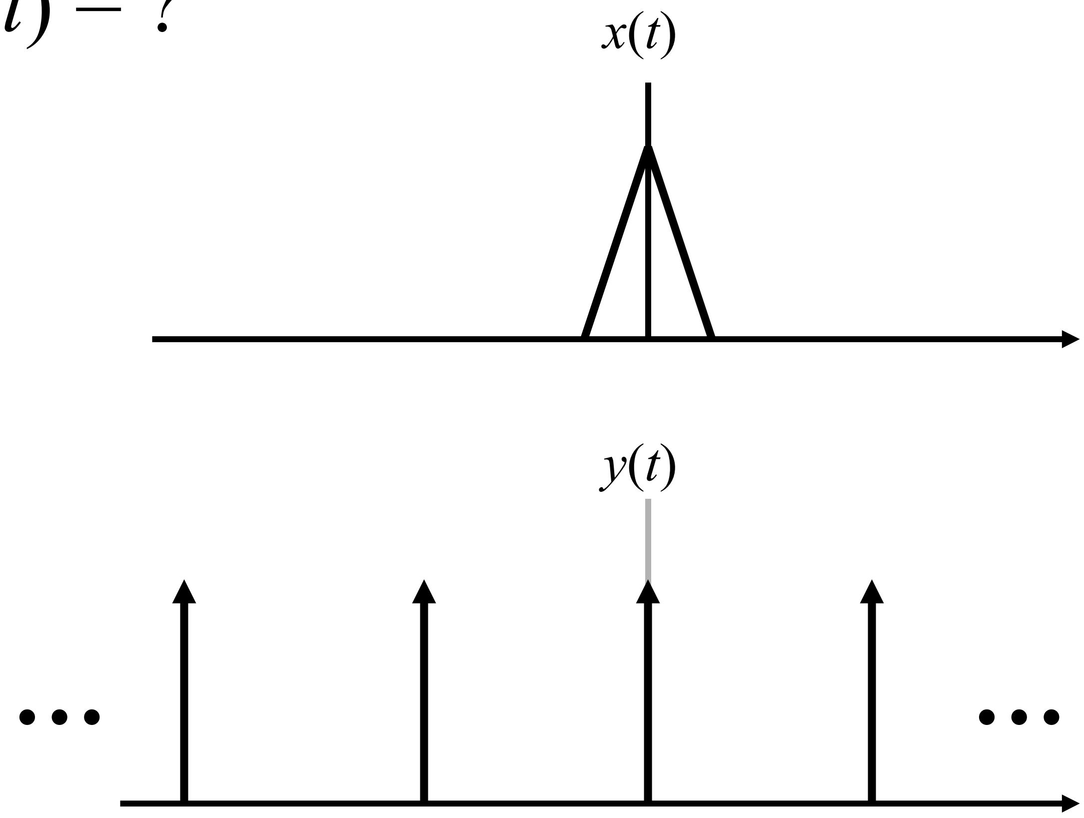
$$x(t) * h(t) = h(t) * x(t)$$

$$x(t) * [h_1(t) * h_2(t)] = [x(t) * h_1(t)] * h_2(t)$$

$$x(t) * [h_1(t) + h_2(t)] = [x(t) * h_1(t)] + [x(t) * h_2(t)]$$

Quiz

- $x(t) * y(t) = ?$



Fourier Transform

$$e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$$

$$x(t) \xrightarrow{F} X(\omega)$$

$$X(\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}\,dt$$

$$X(\omega) \xrightarrow{F^{-1}} x(t)$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(\omega)e^{j\omega t}\,d\omega$$

Properties

$$ax(t)+by(t)\longleftrightarrow aX(\omega)+bY(\omega)$$

$$x(-t)\longleftrightarrow X(-\omega)$$

$$x(at)\longleftrightarrow \frac{1}{|a|}X(\frac{\omega}{a})$$

$$x(t)*y(t)\longleftrightarrow X(\omega)Y(\omega)$$

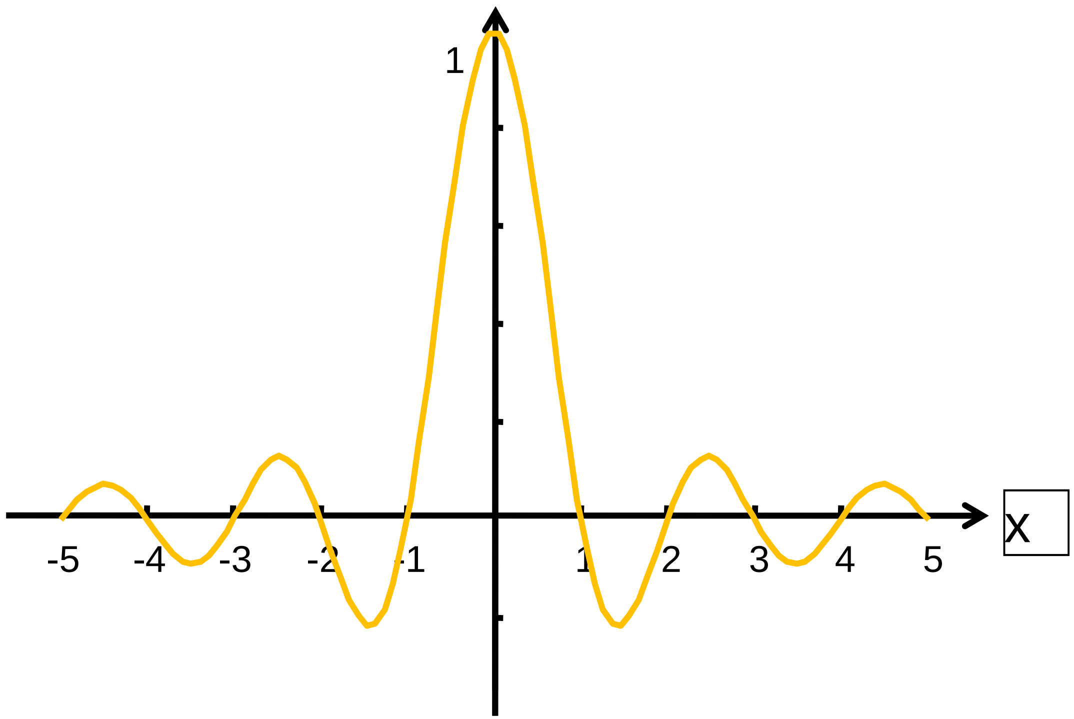
$$x(t)y(t)\longleftrightarrow \frac{1}{2\pi}X(\omega)*Y(\omega)$$

$$\frac{d}{dt}x(t)\longleftrightarrow j\omega X(\omega)$$

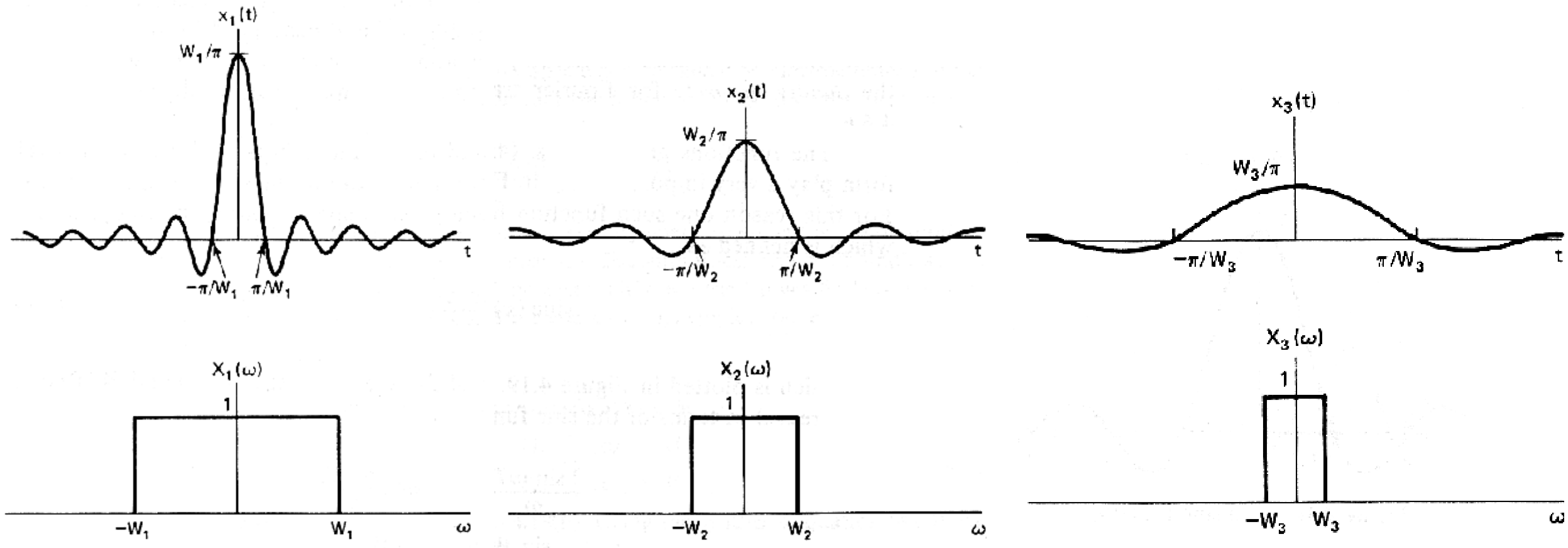
$$\int_{-\infty}^{t}x(t)dt\longleftrightarrow \frac{1}{j\omega}X(\omega)+\pi X(0)\delta(\omega)$$

Sinc function

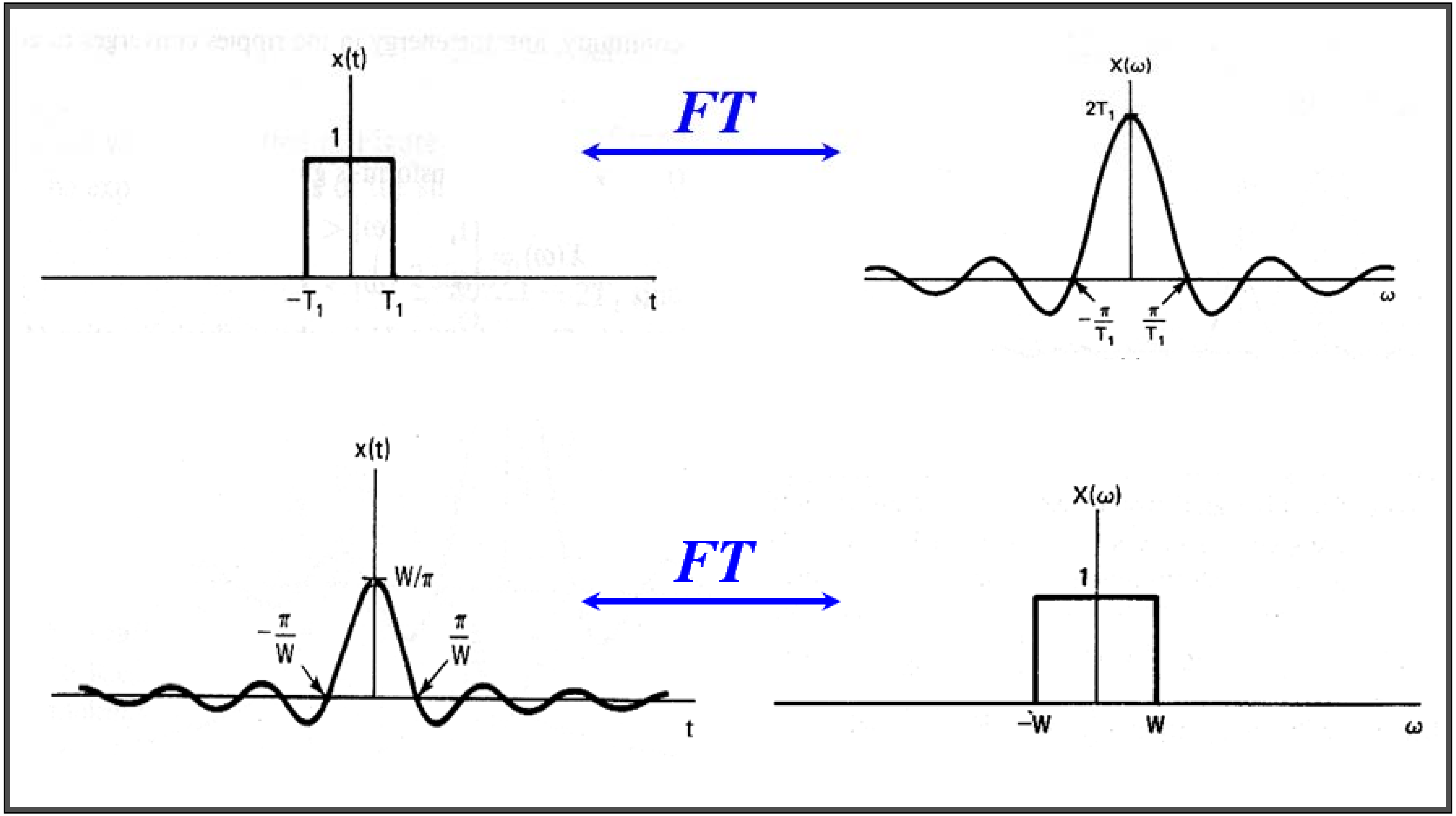
$$\mathrm{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$$



Sinc with different W



Properties: duality



Properties: duality

廣義的
Fourier Transform

$$f(u) = \int_{-\infty}^{+\infty} g(v) e^{-juv} dv$$

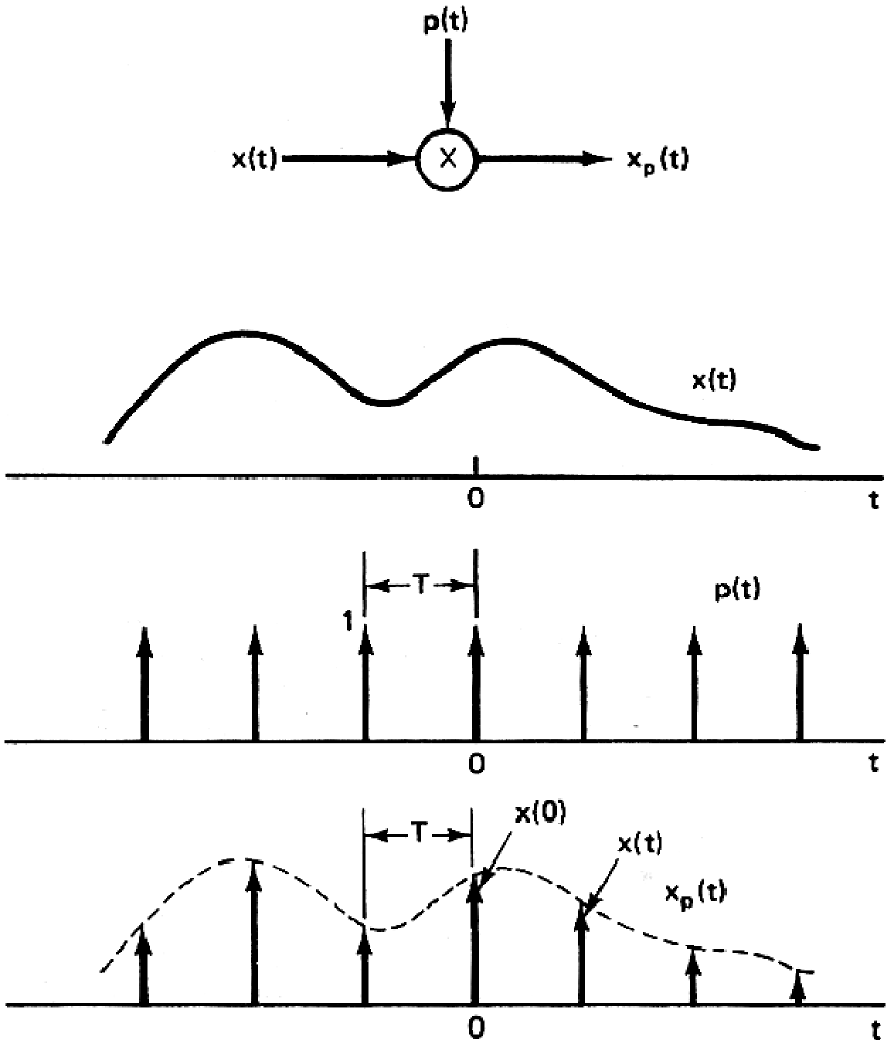
$$g(v) = \int_{-\infty}^{+\infty} f(u) e^{juv} dv$$

$$f(\omega) = \mathrm{F}\{g(t)\}$$
$$g(t) = \mathrm{F}^{-1}\{f(\omega)\}$$

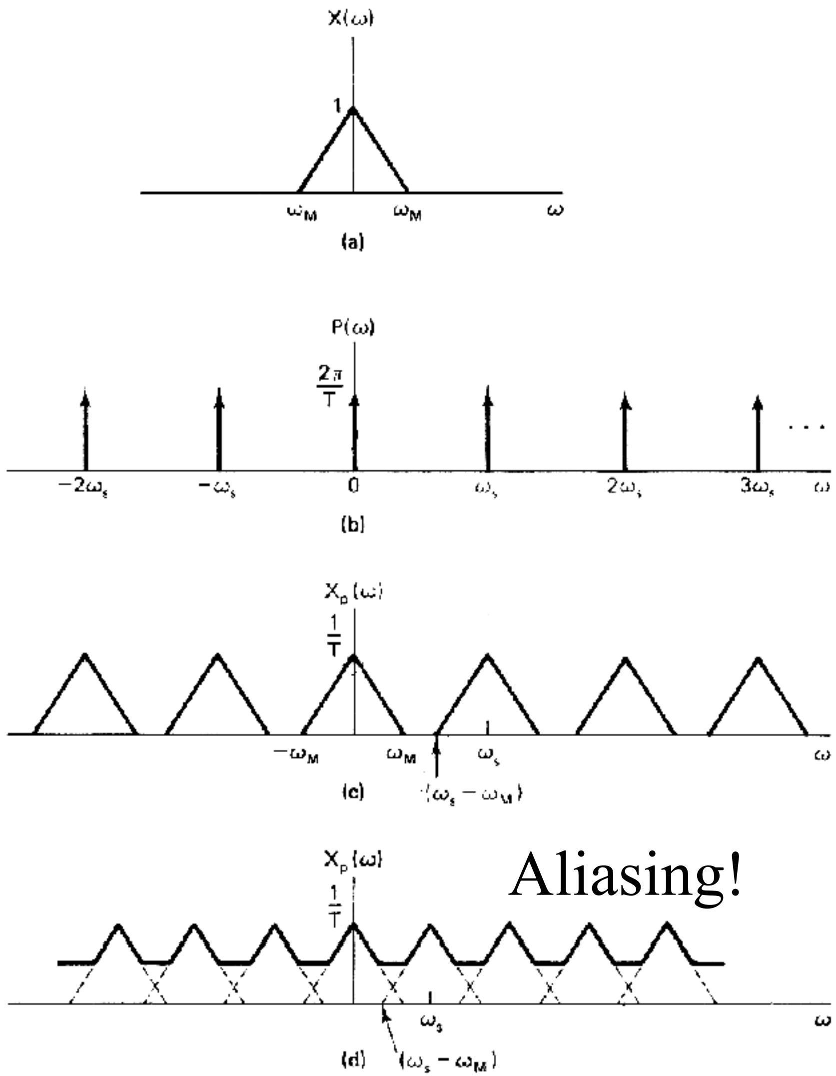
$$\xrightarrow{\text{Duality}}$$

$$\boxed{g(-\omega) = \frac{1}{2\pi} \mathrm{F}\{f(t)\}}$$

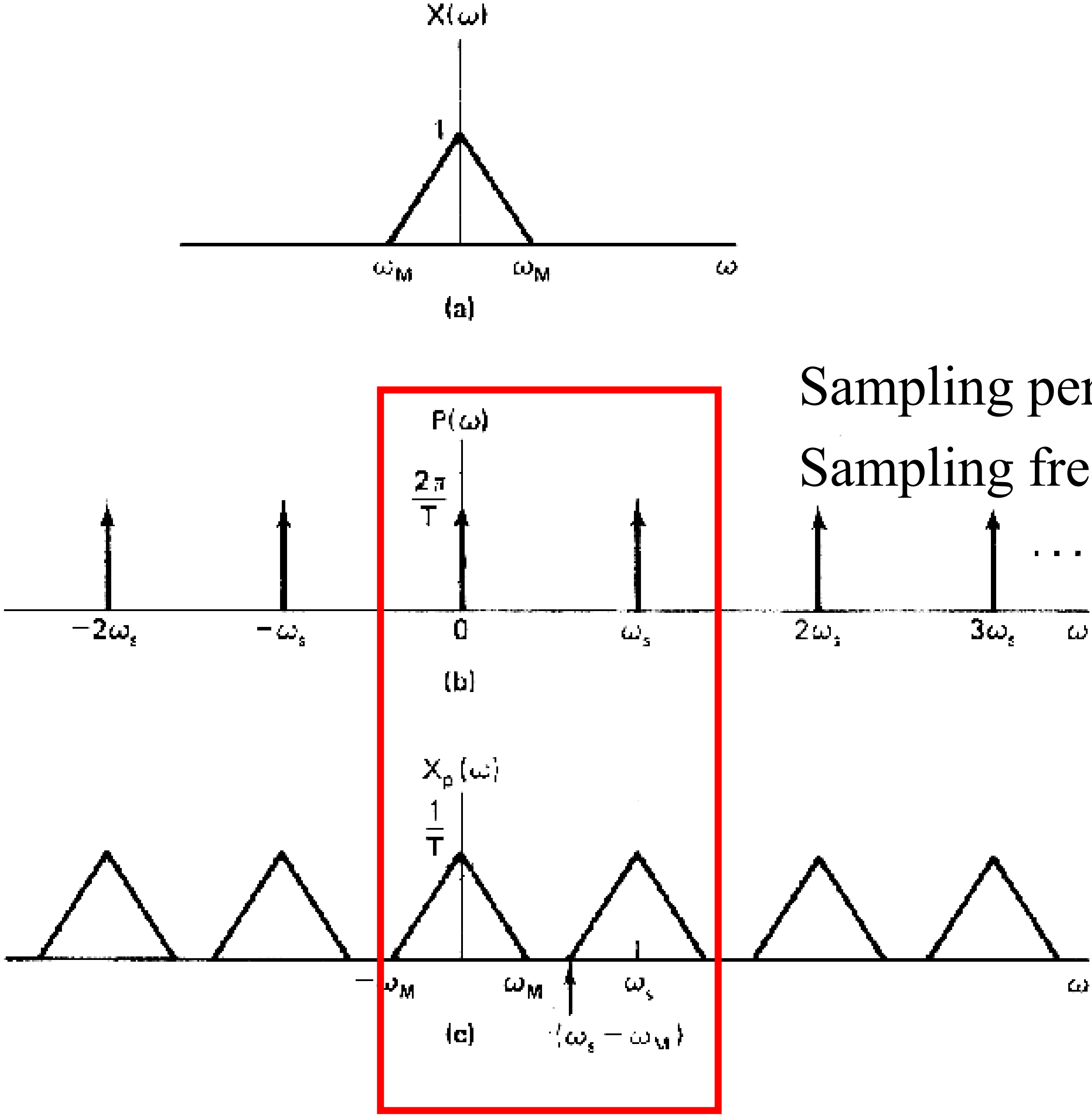
Sampling



p(t)
x(t)
X
$x_p(t)$
x(t)
0
t
T
1
p(t)
0
t
T
x(0)
x(t)
$x_p(t)$
0
t



$X(\omega)$
1
ω_M
ω_M
ω
(a)
$P(\omega)$
$\frac{2\pi}{T}$
$-2\omega_s$
$-\omega_s$
0
ω_s
$2\omega_s$
$3\omega_s$
ω
(b)
$X_p(\omega)$
$\frac{1}{T}$
$-\omega_M$
ω_M
ω_s
ω
(c)
$(\omega_s - \omega_M)$
$X_p(\omega)$
$\frac{1}{T}$
Aliasing!
ω_s
ω
(d)
$(\omega_s - \omega_M)$



Sampling period: T

Sampling frequency: $\omega_s = 2\pi/T$

…

Sampling Theorem

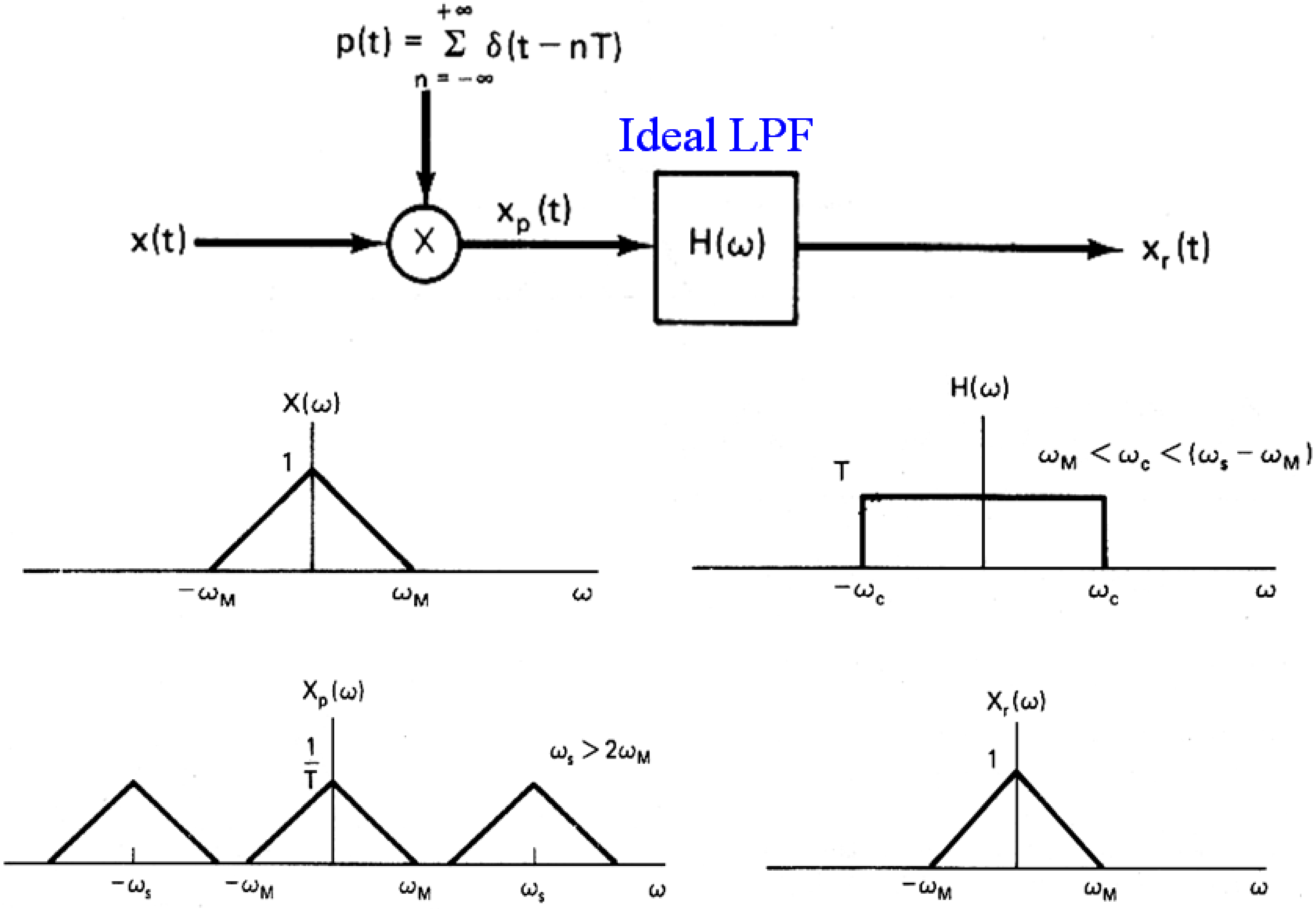
- Let $x(t)$ be a band-limited signal with $X(\omega)=0$ for $|\omega| > \omega_M$,

  then $x(t)$ is uniquely determined by its samples $x(nT)$, $n = 0, \pm 1, \pm 2, \ldots,$ if

$$\boxed{\omega_s > 2\,\omega_M}$$

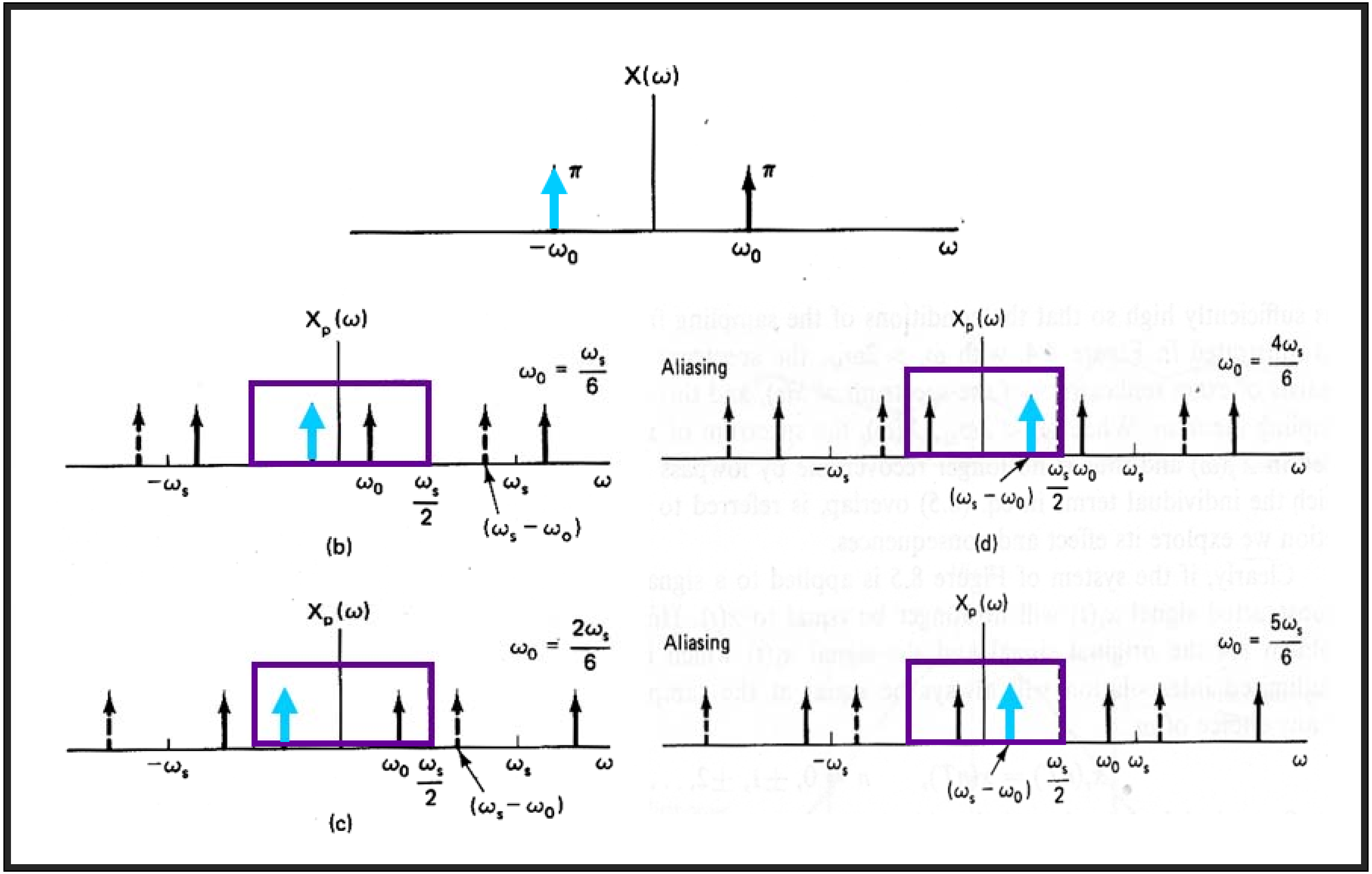
  where

$$\omega_s = \frac{2\pi}{T}$$

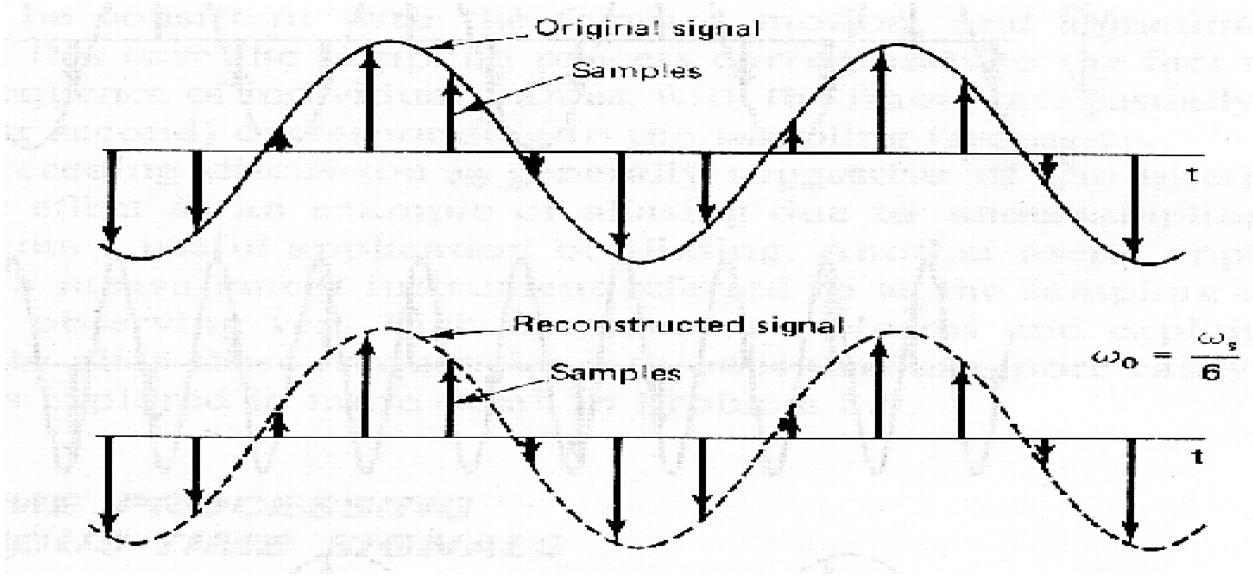


$p(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT)$
Ideal LPF
$x(t)$
$x_p(t)$
$H(\omega)$
$x_r(t)$
$X(\omega)$
1
$-\omega_M$
ω_M
ω
$H(\omega)$
T
$\omega_M < \omega_c < (\omega_s - \omega_M)$
$-\omega_c$
ω_c
ω
$X_p(\omega)$
$\frac{1}{T}$
$\omega_s > 2\omega_M$
$-\omega_s$
$-\omega_M$
ω_M
ω_s
ω
$X_r(\omega)$
1
$-\omega_M$
ω_M
ω

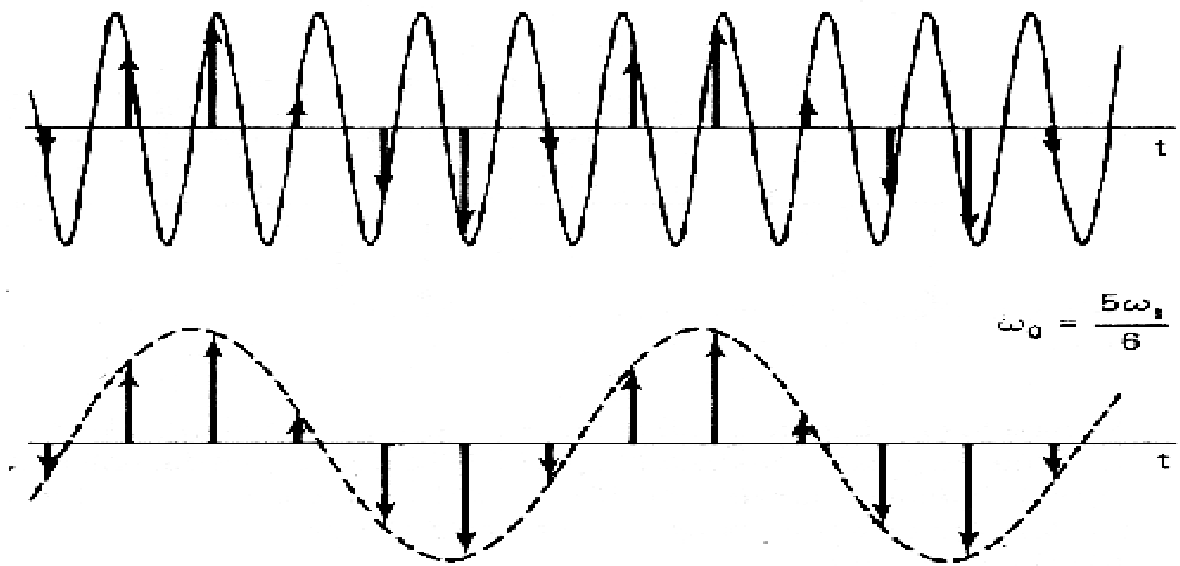
Aliasing



X(ω)
π
π
−ω0
ω0
ω
Xp(ω)
ω0 = ωs/6
−ωs
ω0
ωs/2
ωs
ω
(ωs − ω0)
(b)
Xp(ω)
ω0 = 2ωs/6
−ωs
ω0
ωs/2
ωs
ω
(ωs − ω0)
(c)
Aliasing
Xp(ω)
ω0 = 4ωs/6
−ωs
ωs/2
ω0
ωs
ω
(ωs − ω0)
(d)
Aliasing
Xp(ω)
ω0 = 5ωs/6
−ωs
ωs/2
ω0
ωs
ω
(ωs − ω0)



Original signal
Samples
t
Reconstructed signal
Samples
$\omega_0 = \frac{\omega_s}{6}$
t



t
$\omega_0 = \frac{5\omega_s}{6}$
t

Continuous-time Fourier Transform

$$e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$$

$$x(t) \xrightarrow{F} X(\omega)$$

$$X(\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$$

$$X(\omega) \xrightarrow{F^{-1}} x(t)$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(\omega) e^{j\omega t} d\omega$$

Discrete-time Fourier Transform

$$X(\Omega) = \sum_{-\infty}^{+\infty} x[n] e^{-j\Omega n}$$

- Discrete in time domain
- Continuous and periodical in frequency domain
- $\Omega = -\pi \sim +\pi$ corresponds to $-f_s/2 \sim +f_s/2$

How to create the discrete form in the frequency domain?

Periodic $$X(\Omega) = \sum_{-\infty}^{+\infty} x[n] e^{-j\Omega n}$$

Discrete $$x[n] = \frac{1}{2\pi} \int_{-\pi}^{+\pi} X(\Omega) e^{j\Omega n} d\Omega$$

What if x is discrete and periodic?

Discrete Fourier Transform (DFT)

Both $x[n]$ and $X[k]$ are periodic with a cycle of N points, and $\Omega = \frac{2\pi k}{N}$.

$$X[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-jk(2\pi/N)n}, k = 0,1,\ldots, N-1$$

$$x[n] = \sum_{k=0}^{N-1} X[k]e^{jk(2\pi/N)n}, n = 0,1,\ldots, N-1$$

延伸一下

- 剛剛說的東西通通都可以推廣到2D
  - Delta function: $\delta\,(x, y)$
  - Convolution

$$g(x,y)=\int_{-\infty}^{+\infty} f(\delta,\eta)h(x-\delta,y-\eta)d\delta d\eta$$

  - Sampling theorem
  - Fourier transform

2D - Fourier Transform

$$X(\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$$

ω_x , ω_y $\quad\Big\downarrow\quad$ $f(x, y)$ $\quad$ dx, dy

$$F(\omega_x, \omega_y) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x, y) e^{-j\omega_x x} e^{-j\omega_y y} dx dy$$

$$f(x, y) = \frac{1}{(2\pi)^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} F(\omega_x, \omega_y) e^{j\omega_x x} e^{j\omega_y y} d\omega_x d\omega_y$$

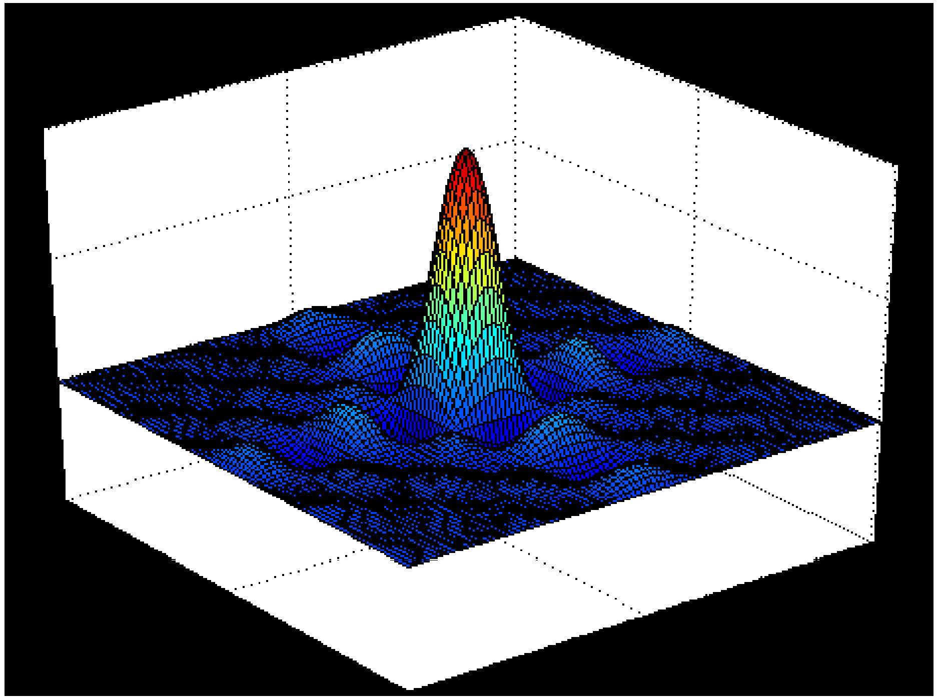
Properties of 2D-FT

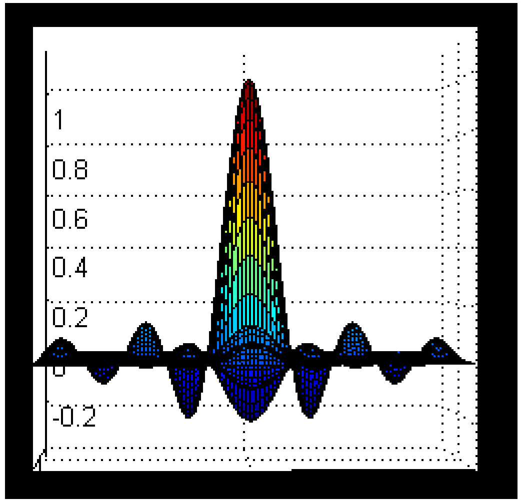
	Spatial Domain†		Frequency Domain†
1)	$f(x, y)$ real	$\Leftrightarrow$	$F^*(u, v) = F(-u, -v)$
2)	$f(x, y)$ imaginary	$\Leftrightarrow$	$F^*(-u, -v) = -F(u, v)$
3)	$f(x, y)$ real	$\Leftrightarrow$	$R(u, v)$ even; $I(u, v)$ odd
4)	$f(x, y)$ imaginary	$\Leftrightarrow$	$R(u, v)$ odd; $I(u, v)$ even
5)	$f(-x, -y)$ real	$\Leftrightarrow$	$F^*(u, v)$ complex
6)	$f(-x, -y)$ complex	$\Leftrightarrow$	$F(-u, -v)$ complex
7)	$f^*(x, y)$ complex	$\Leftrightarrow$	$F^*(-u - v)$ complex
8)	$f(x, y)$ real and even	$\Leftrightarrow$	$F(u, v)$ real and even
9)	$f(x, y)$ real and odd	$\Leftrightarrow$	$F(u, v)$ imaginary and odd
10)	$f(x, y)$ imaginary and even	$\Leftrightarrow$	$F(u, v)$ imaginary and even
11)	$f(x, y)$ imaginary and odd	$\Leftrightarrow$	$F(u, v)$ real and odd
12)	$f(x, y)$ complex and even	$\Leftrightarrow$	$F(u, v)$ complex and even
13)	$f(x, y)$ complex and odd	$\Leftrightarrow$	$F(u, v)$ complex and odd

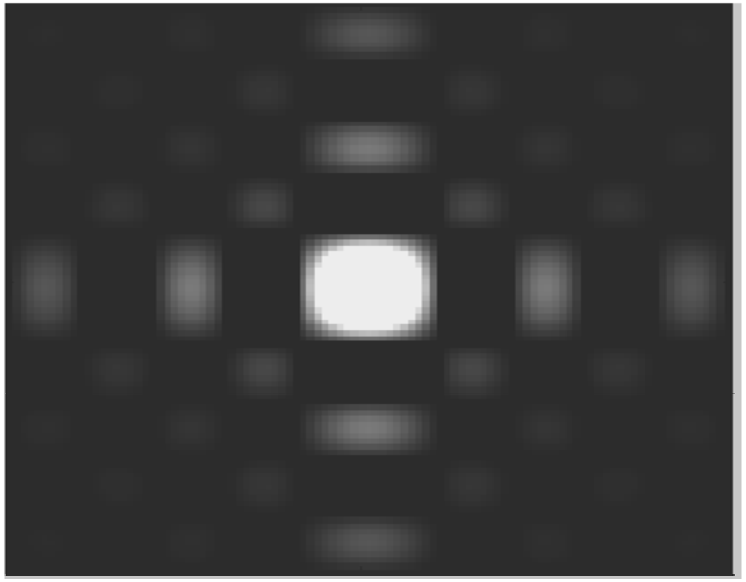
†Recall that x, y, u, and v are *discrete* (integer) variables, with x and u in the range $[0, M - 1]$, and y, and v in the range $[0, N - 1]$. To say that a complex function is *even* means that its real *and* imaginary parts are even, and similarly for an odd complex function.

2D sinc function:

$$F(u,v) = \frac{\sin(\pi u)}{\pi u} \frac{\sin(\pi v)}{\pi v}$$

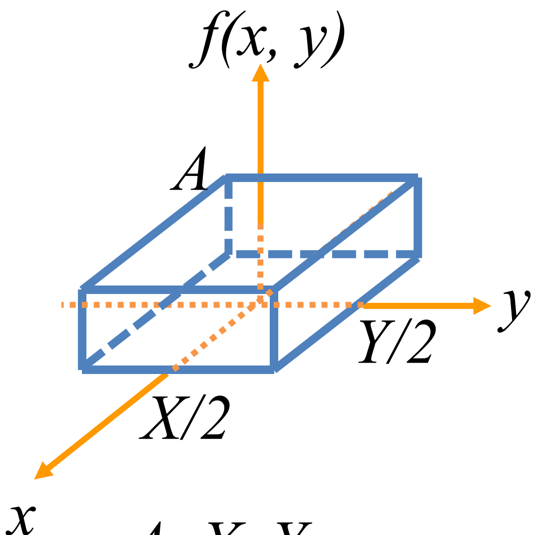




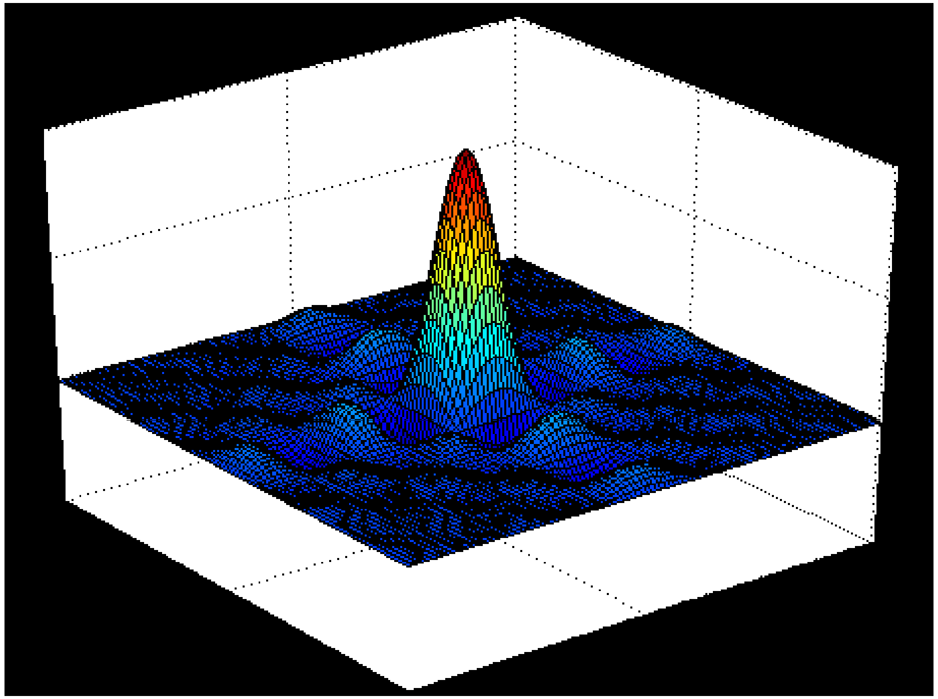


2D – sinc function

$$F(u,v) = AXY\left(\frac{\sin(\frac{uX}{2})}{\frac{uX}{2}}\right)\left(\frac{\sin(\frac{vY}{2})}{\frac{vY}{2}}\right)$$

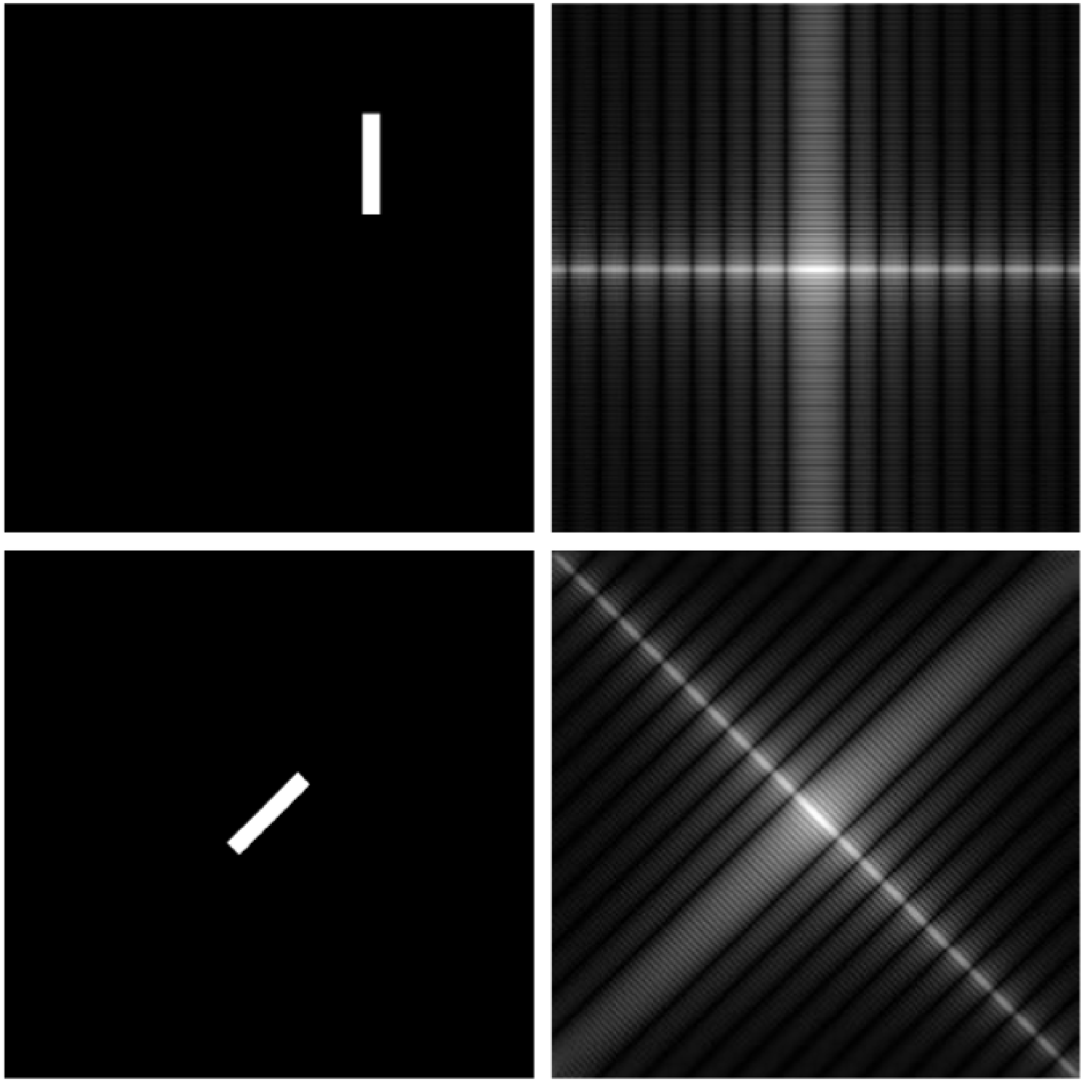


2D-FT



A, X, Y are constants.

Rectangles in space



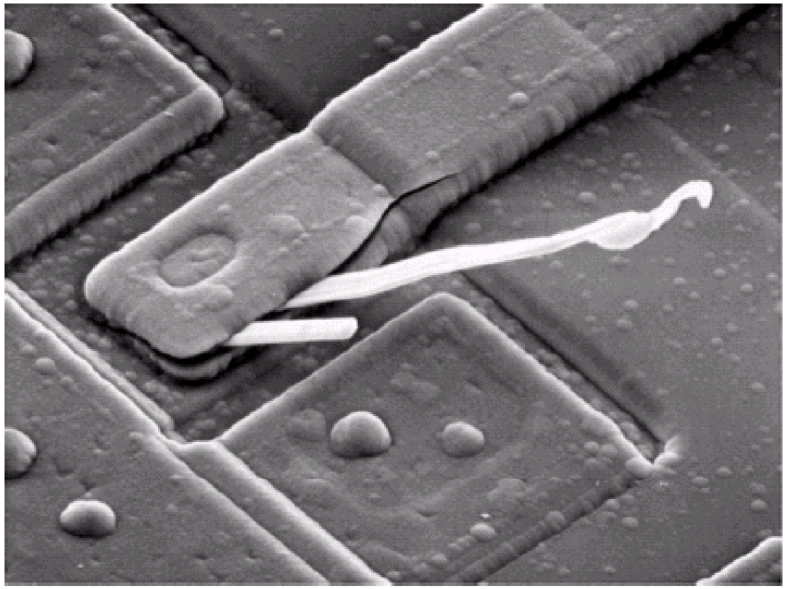
a	b
c	d

FIGURE 4.25 (a) The rectangle in Fig. 4.24(a) translated, and (b) the corresponding spectrum. (c) Rotated rectangle, and (d) the corresponding spectrum. The spectrum corresponding to the translated rectangle is identical to the spectrum corresponding to the original image in Fig. 4.24(a).

2D Fourier transform

Low-freq. → slowly varying

High-freq. → edges, or noise



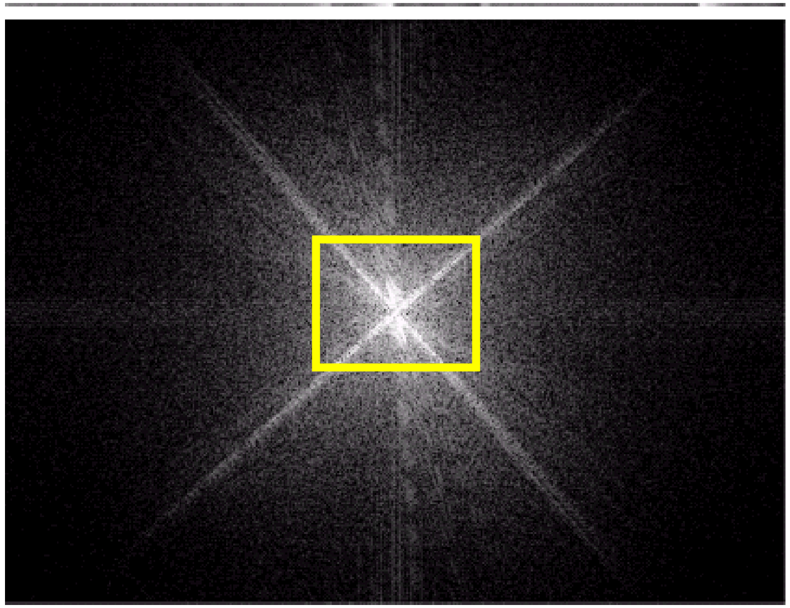
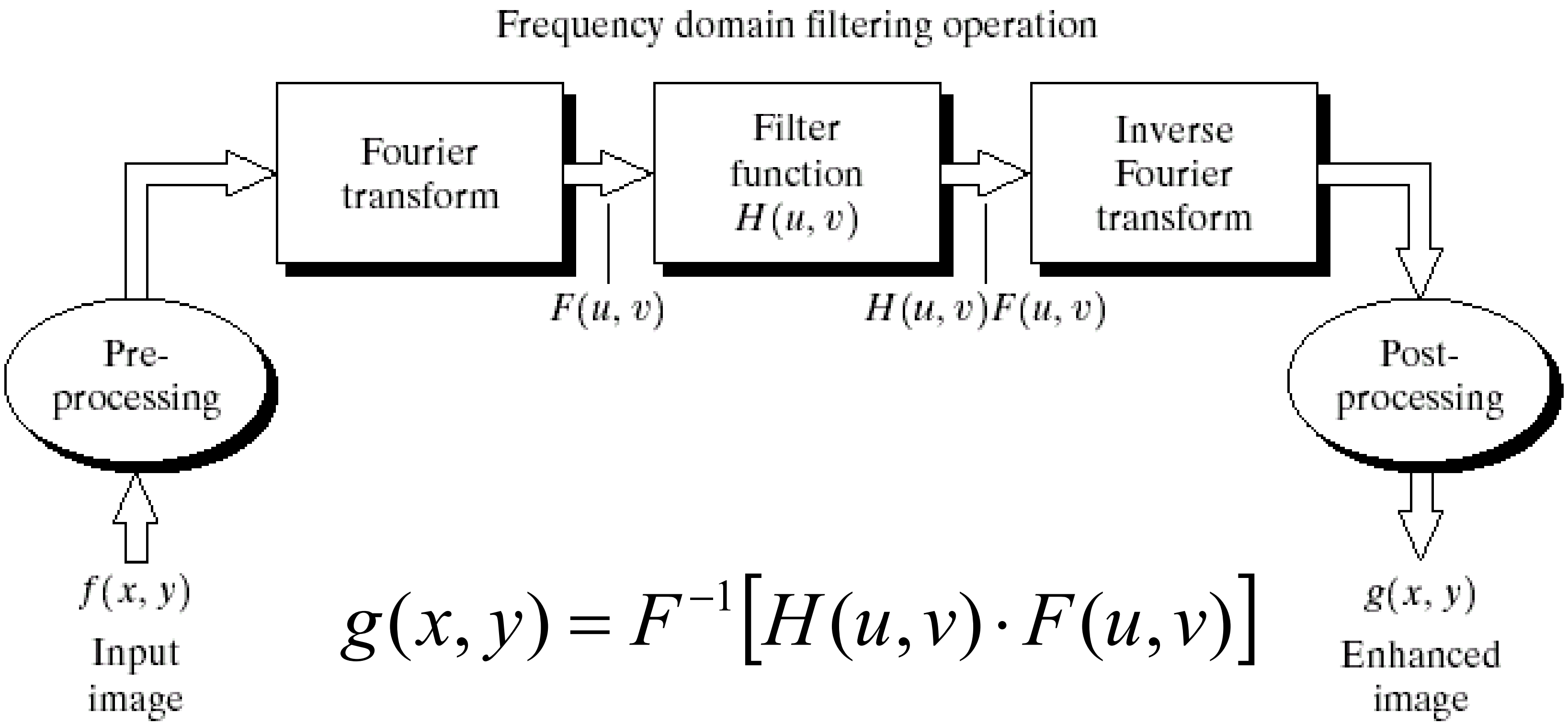


Image Enhancement in the Frequency Domain

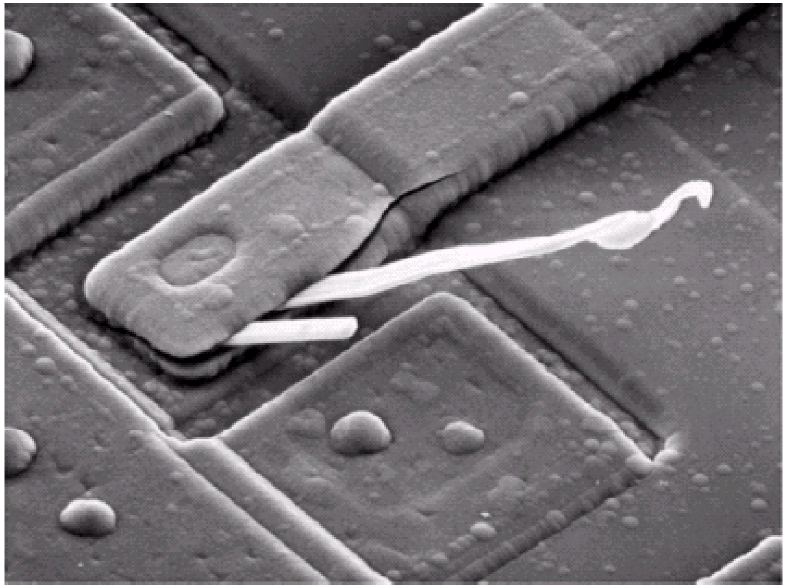
Basic scheme of filtering

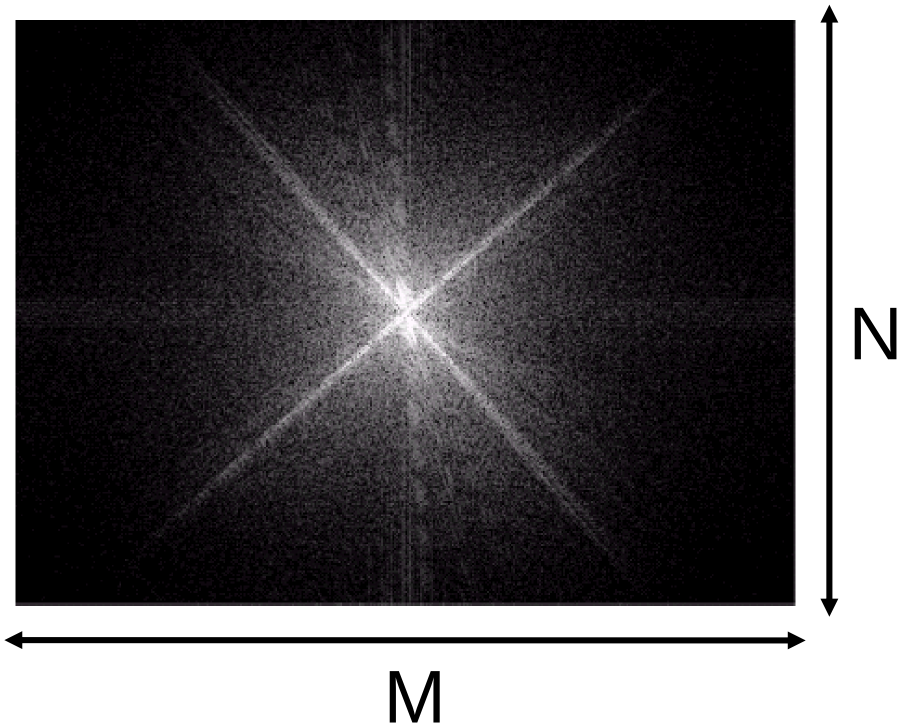


$$g(x,y) = F^{-1}\left[H(u,v)\cdot F(u,v)\right]$$

Basic filter

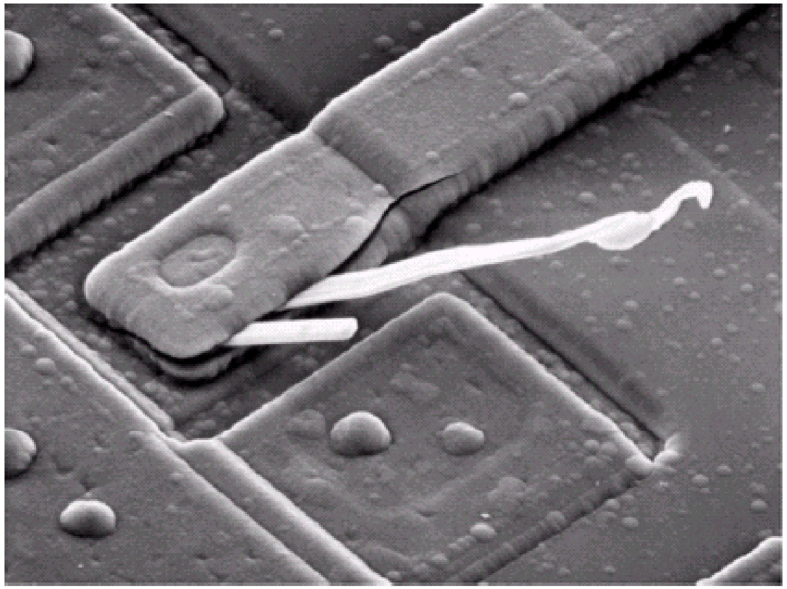
$$H(u,v)=\begin{cases}0, & \text{if } (u,v)=(M/2, N/2)\\ 1, & \text{otherwise.}\end{cases}$$

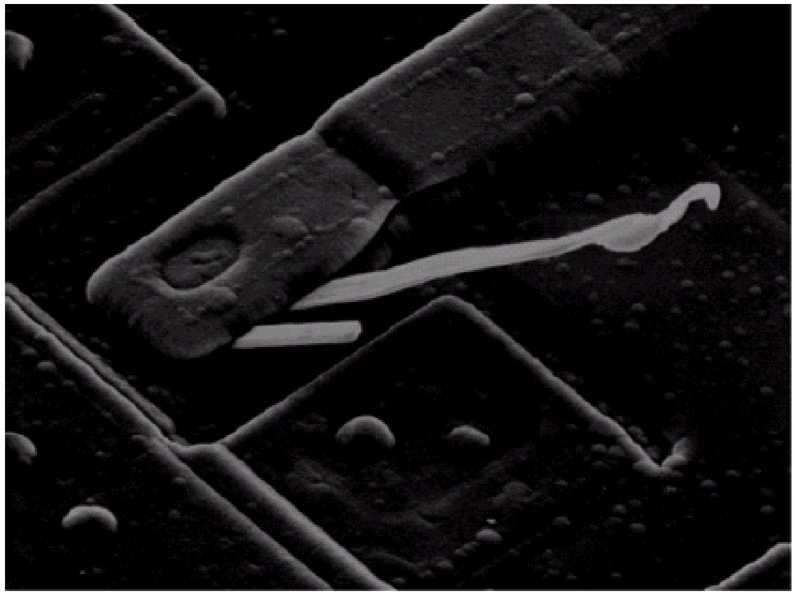




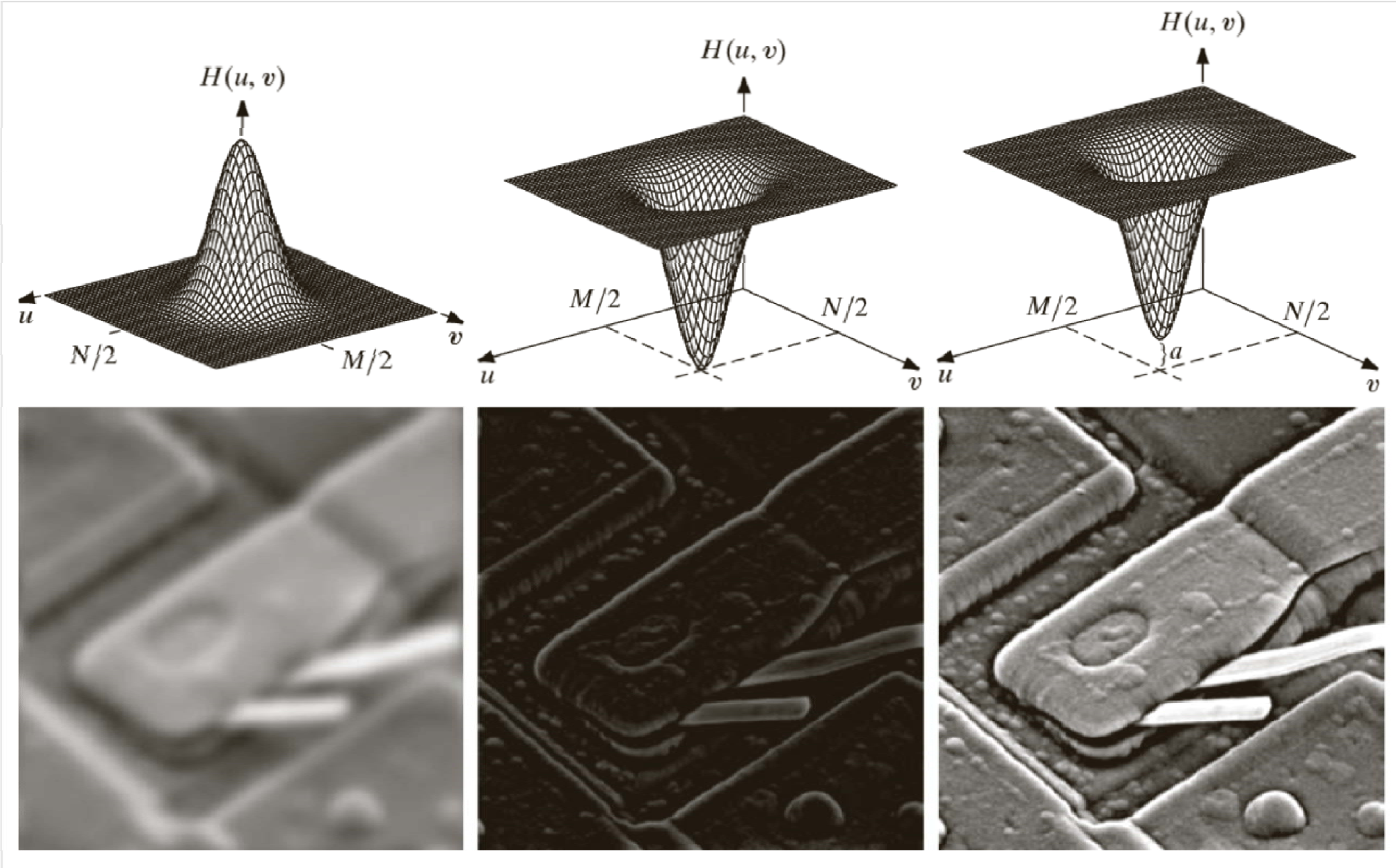
Basic filter: DC rejection

$$H(u,v)=\begin{cases}0, & if\,(u,v)=(M/2,N/2)\\1, & otherwise.\end{cases}$$



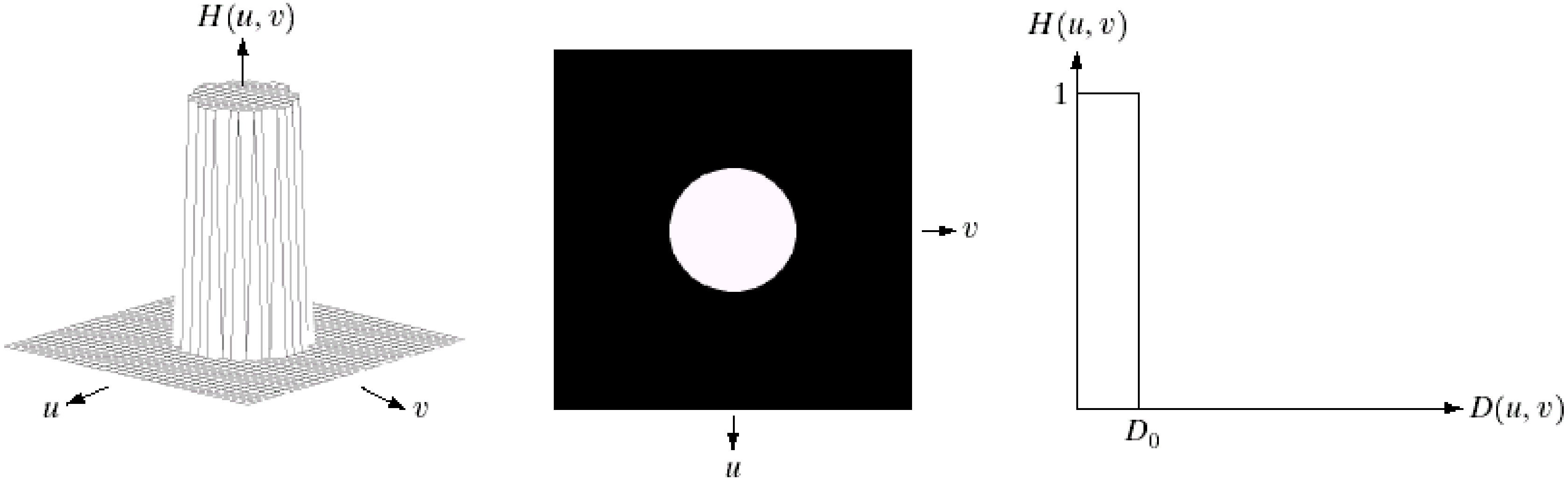


Filtering



Ideal Low-pass Filter

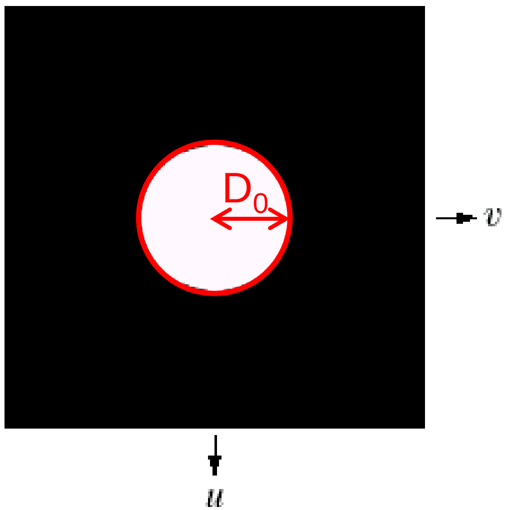
$$H(u,v) = \begin{cases} 1, & if \quad D(u,v) \le D_0 \\ 0, & if \quad D(u,v) > D_0 \end{cases}$$

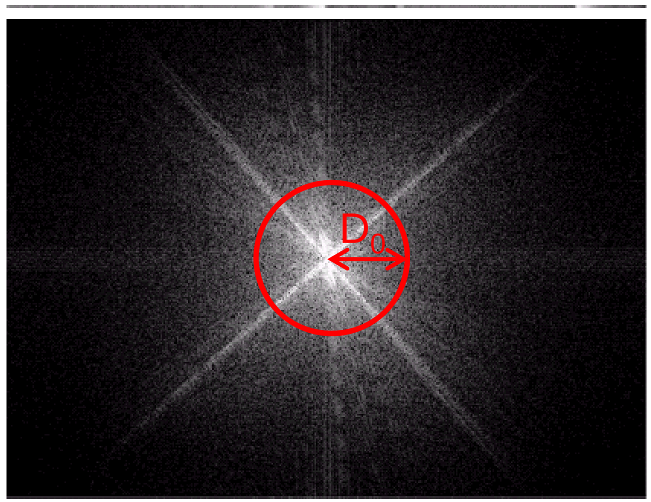


$$D(u,v) = [(u - M/2)^2 + (v - N/2)^2]^{1/2}$$

Percentage of Power

- $D_0 \rightarrow$ cutoff frequency
- Percentage of power: $\alpha = 100\left[\sum_u \sum_v P(u,v) / P_T\right]$





Percentage of Power

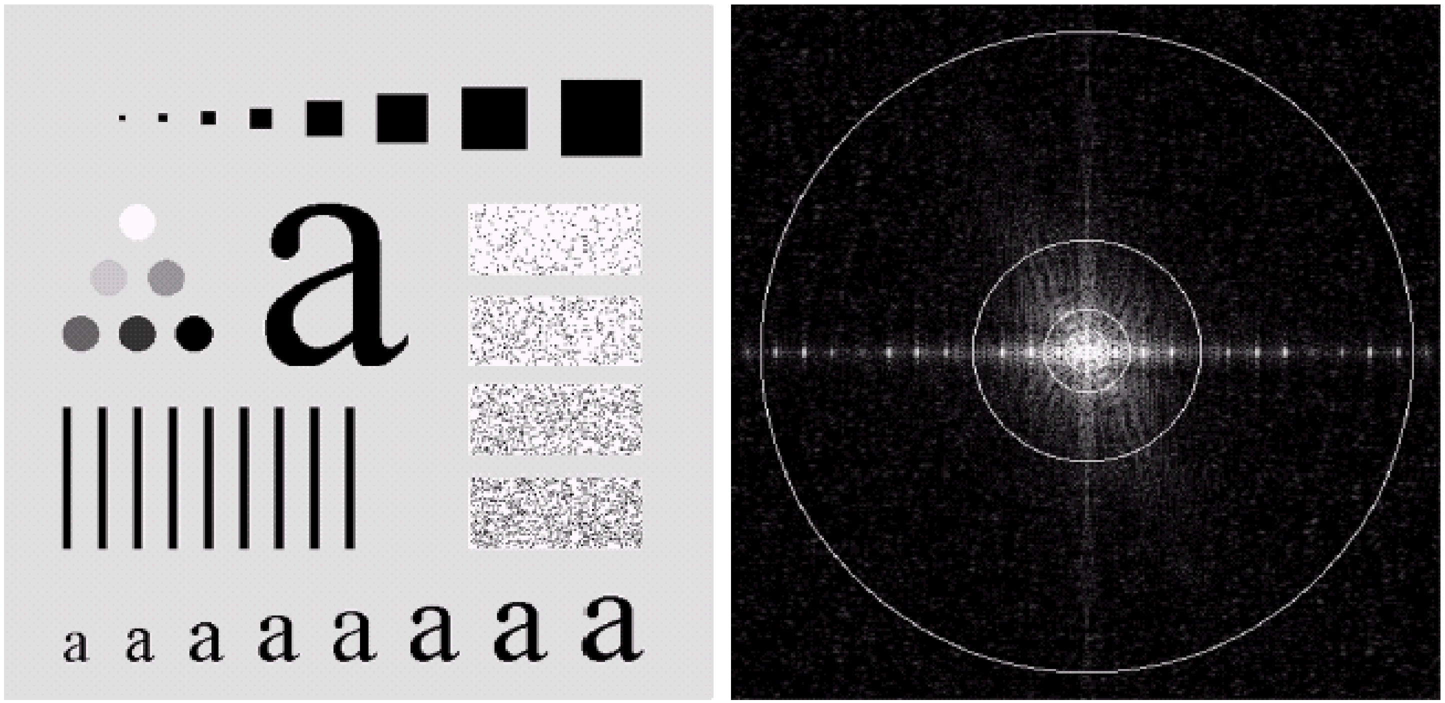
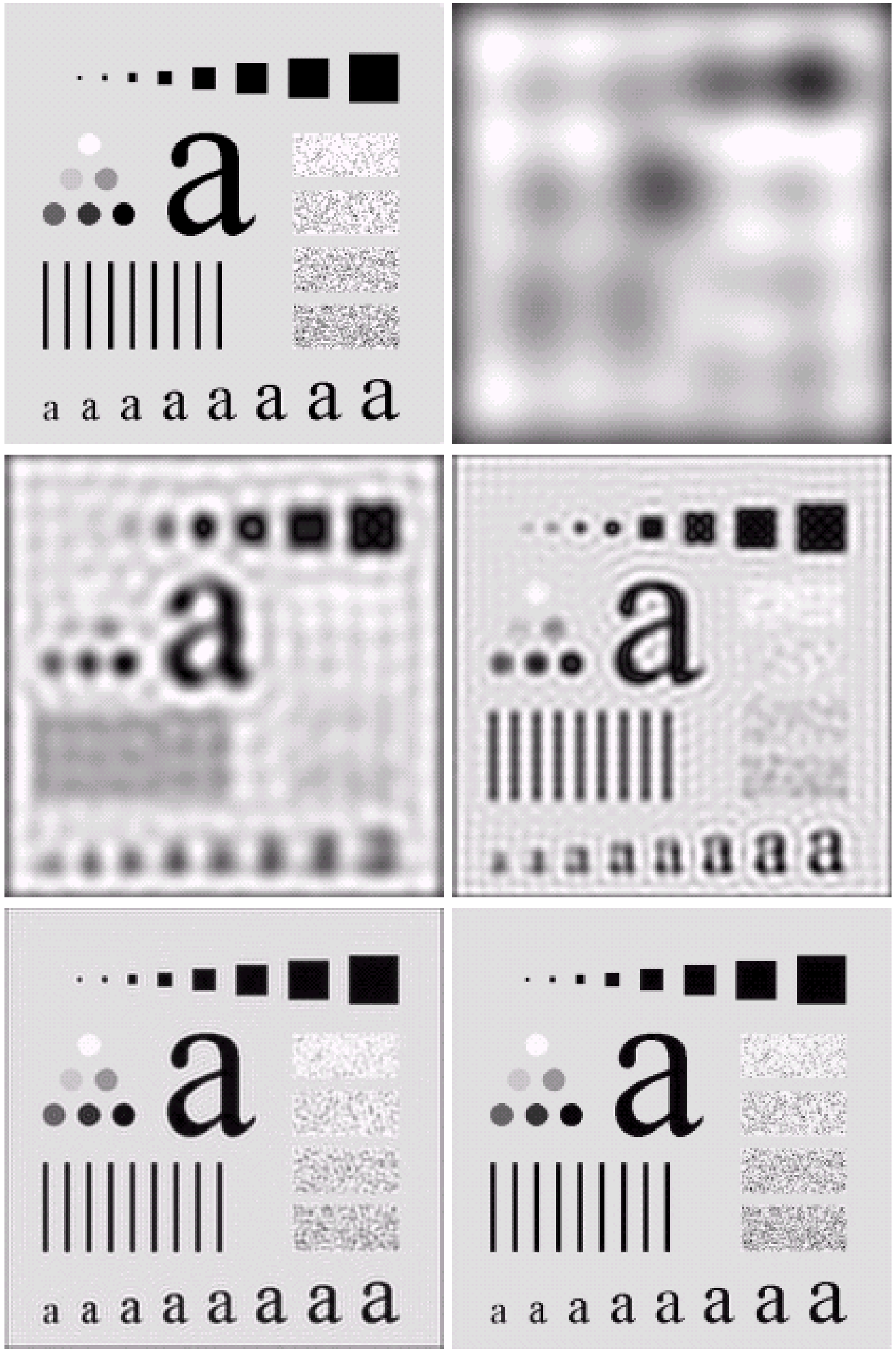


FIGURE 4.11 (a) An image of size 500 × 500 pixels and (b) its Fourier spectrum. The superimposed circles have radii values of 5, 15, 30, 80, and 230, which enclose 92.0, 94.6, 96.4, 98.0, and 99.5% of the image power, respectively.

Notice the “ringing” in b-f



(a) original

(b) D_0 = 5

(d) D_0 = 15

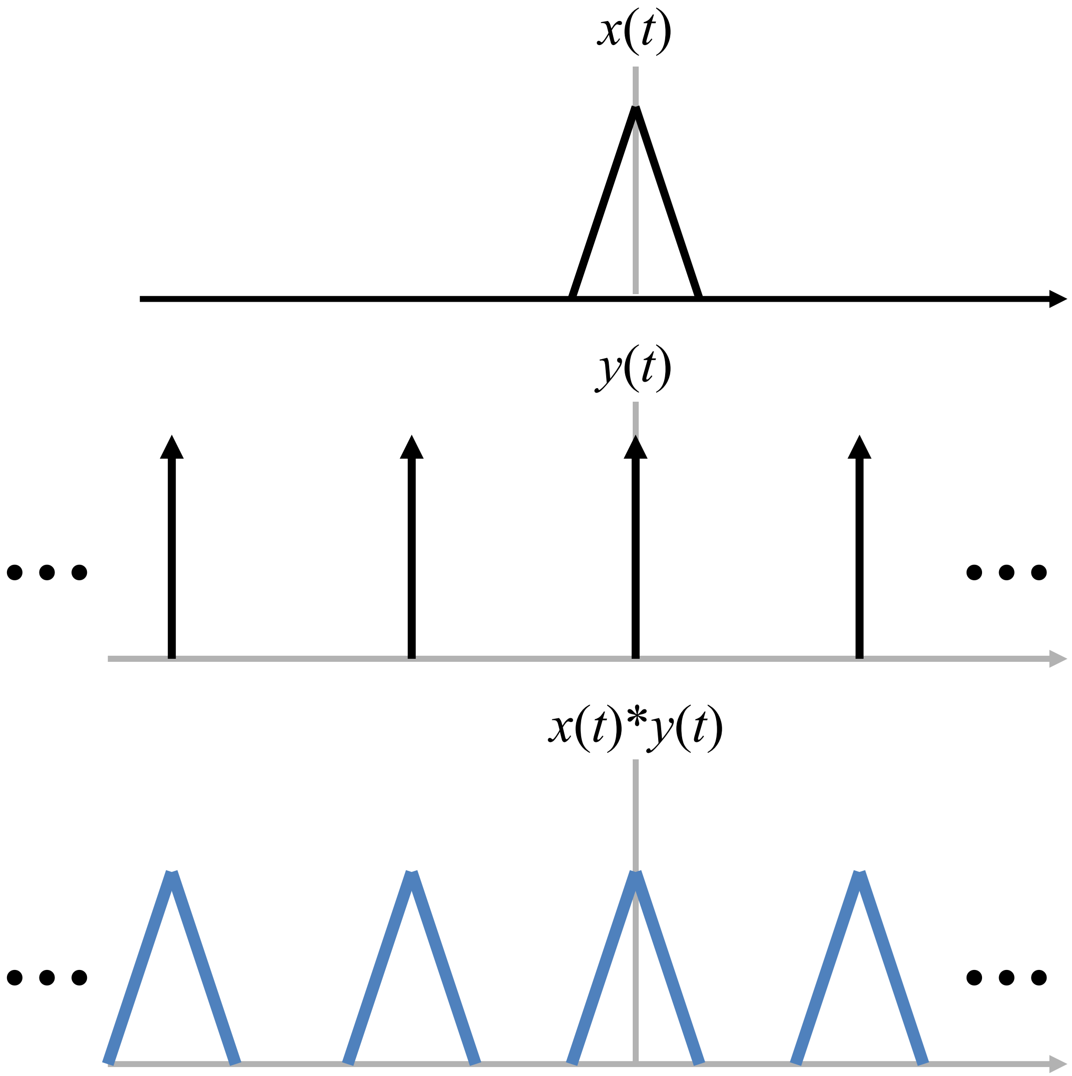
(d) D_0 = 30

(f) D_0 = 80

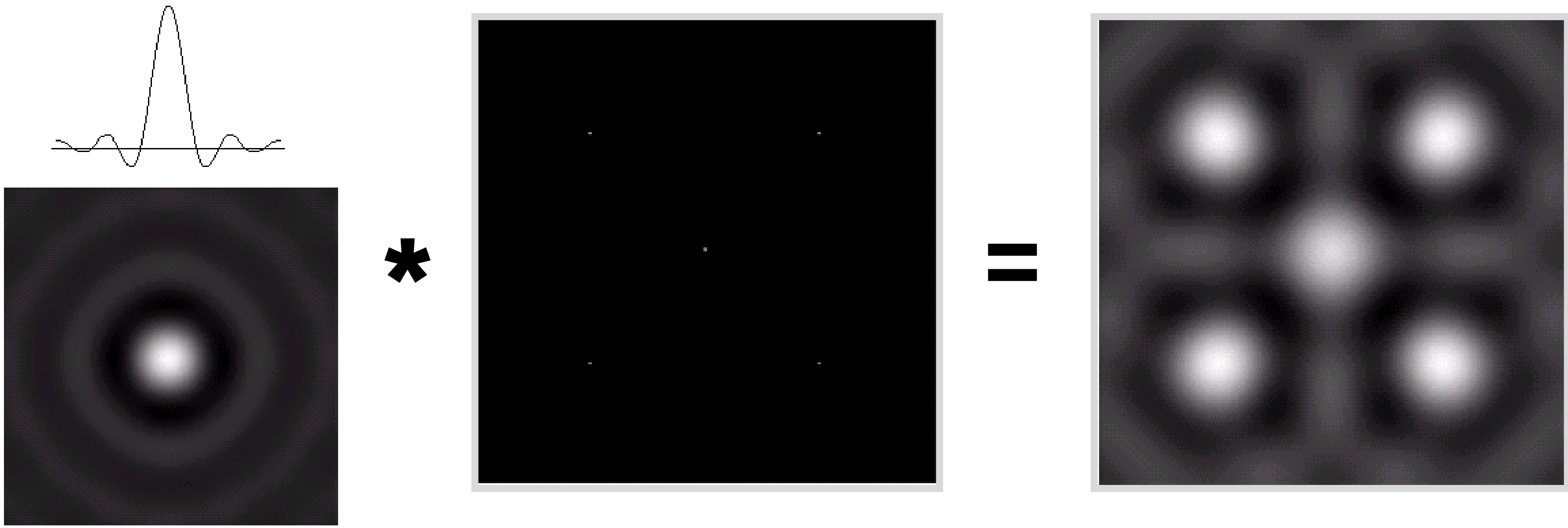
(f) D_0 = 230

Recall: convolution

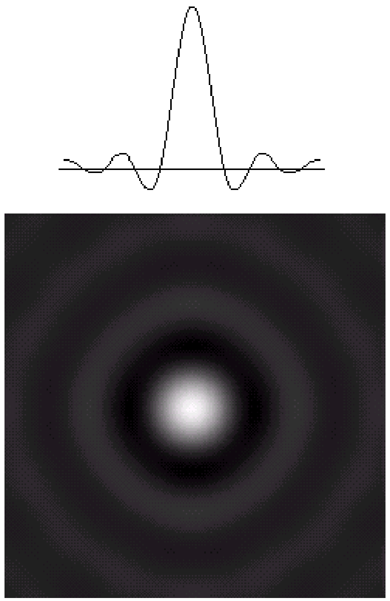
- $x(t) * y(t) =$



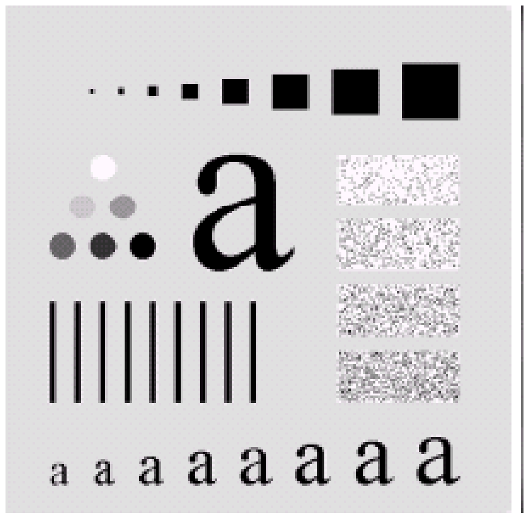
2D convolution



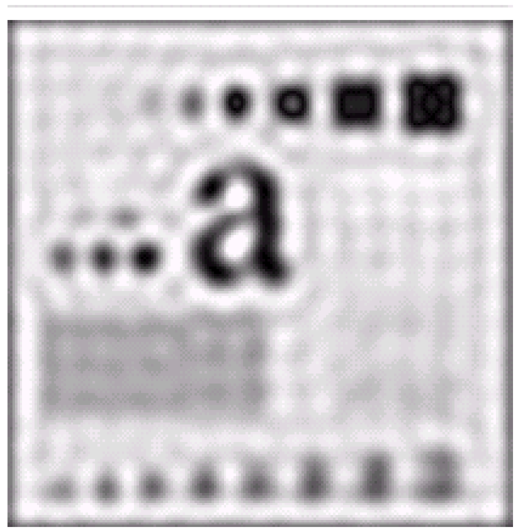
2D convolution



*



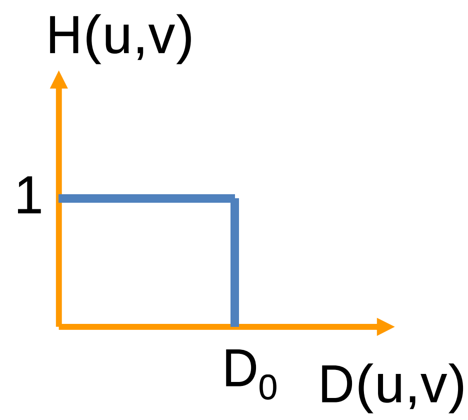
=



Butterworth Lowpass filter

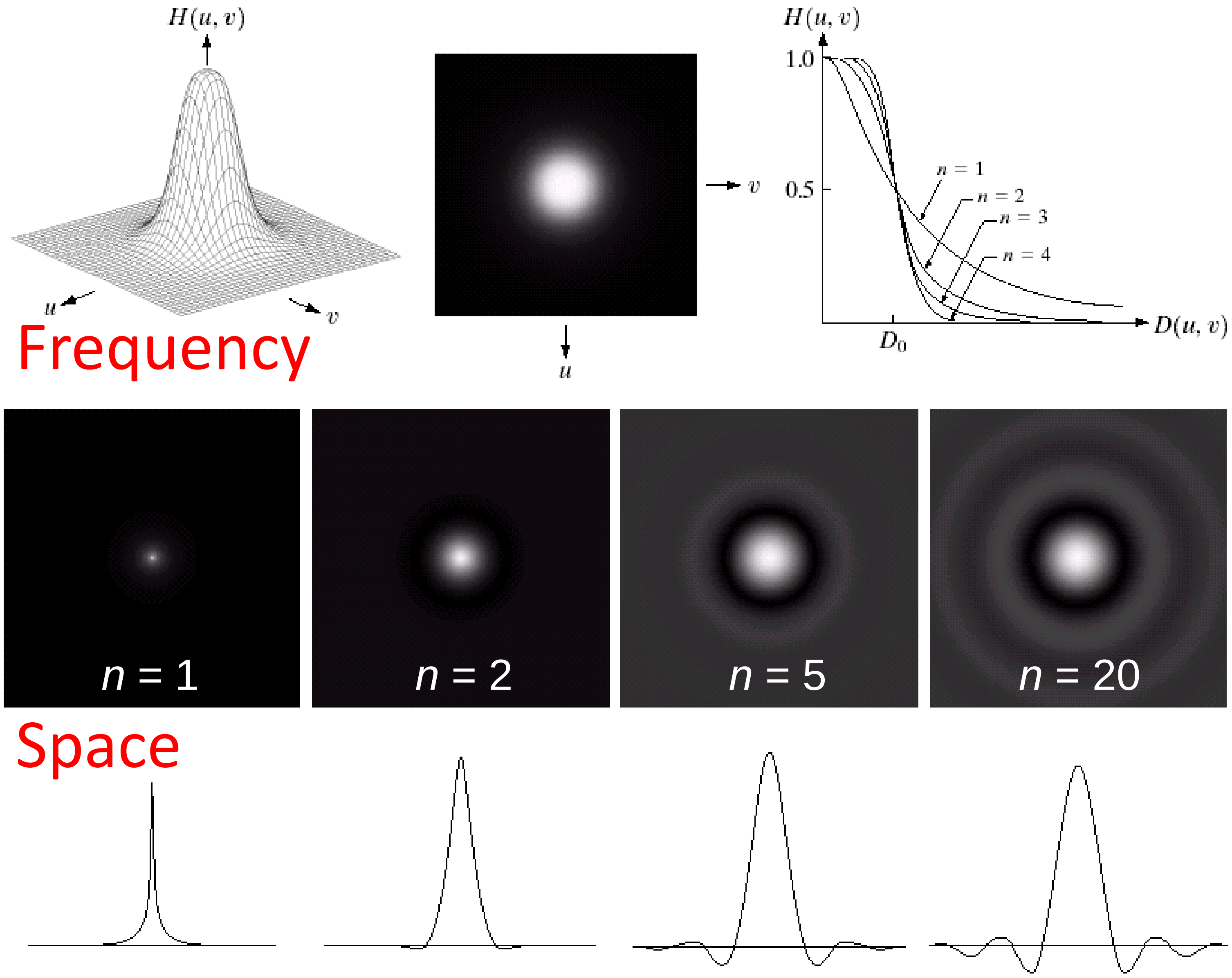
- ILPF $$G(u,v) = H(u,v) \cdot F(u,v)$$

$$H(u,v) = \begin{cases} 1, & if \quad D(u,v) \le D_0 \\ 0, & if \quad D(u,v) > D_0 \end{cases}$$



- Butterworth

$$H(u,v) = \frac{1}{1 + \left[D(u,v) / D_0 \right]^{2n}}$$

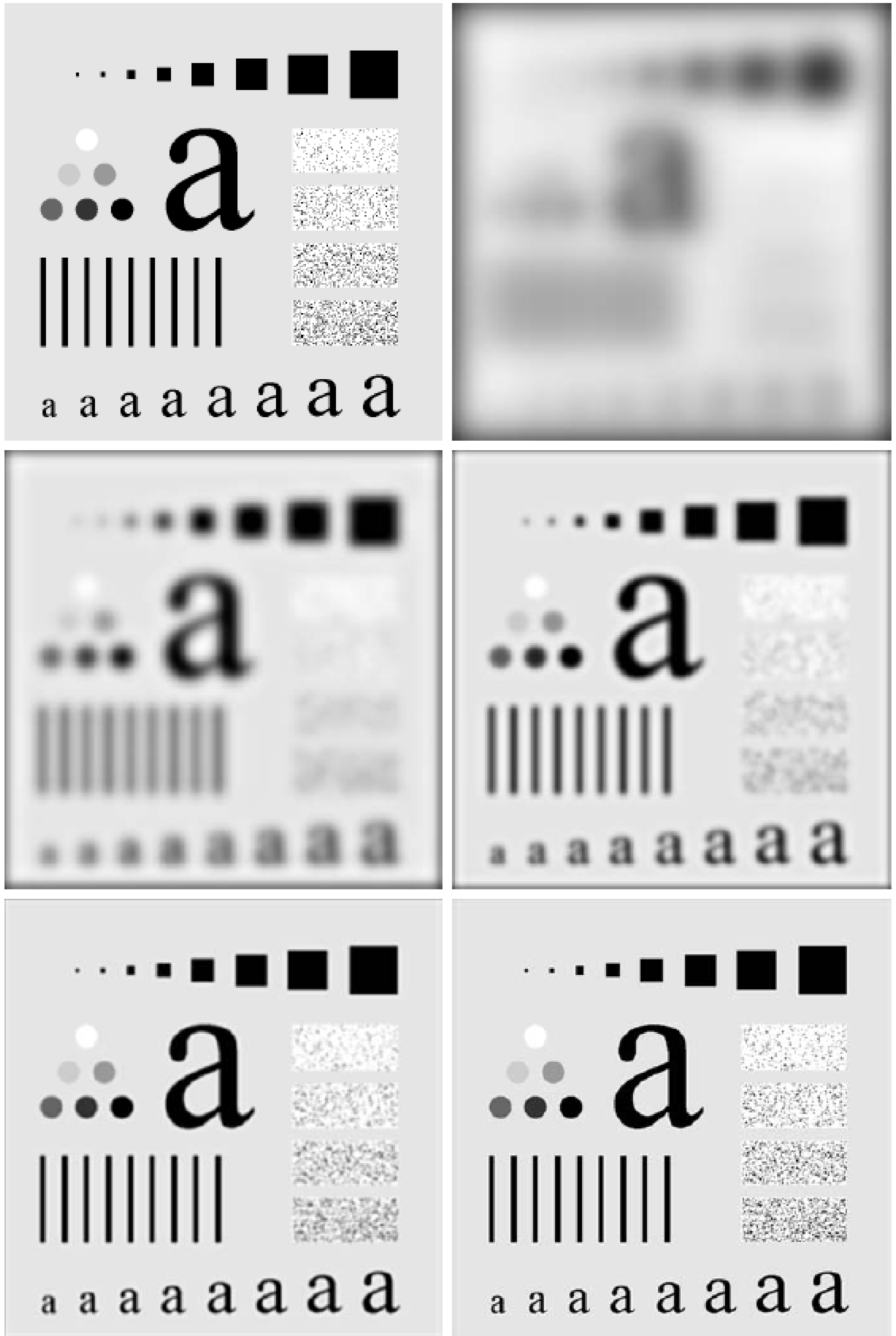


a b c d

FIGURE 4.16 (a)–(d) Spatial representation of BLPFs of order 1, 2, 5, and 20, and corresponding gray-level profiles through the center of the filters (all filters have a cutoff frequency of 5). Note that ringing increases as a function of filter order.

Butterworth LPF
(n = 2)

Ringing disappear!



(a) original

(b) $D_0 = 5$

(d) $D_0 = 15$

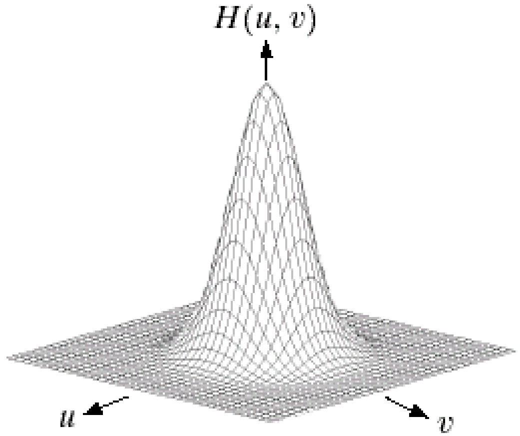
(d) $D_0 = 30$

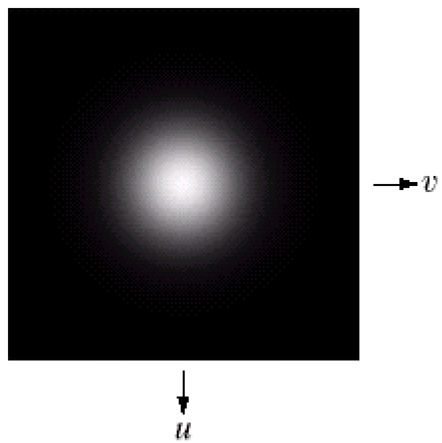
(f) $D_0 = 80$

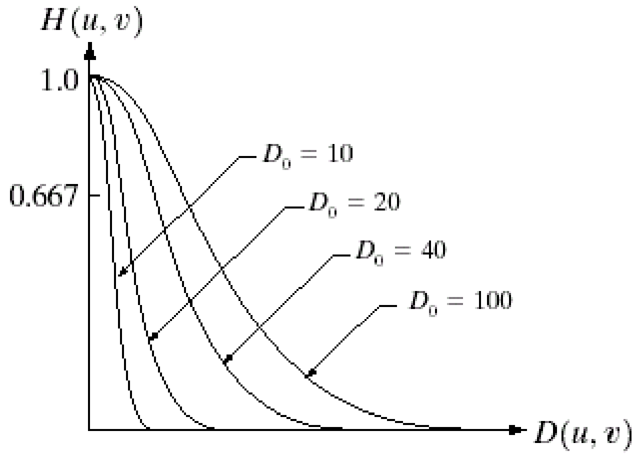
(f) $D_0 = 230$

Gaussian Lowpass Filter

$$H(u,v) = \exp(-D^2(u,v)/2D_0{}^2)$$



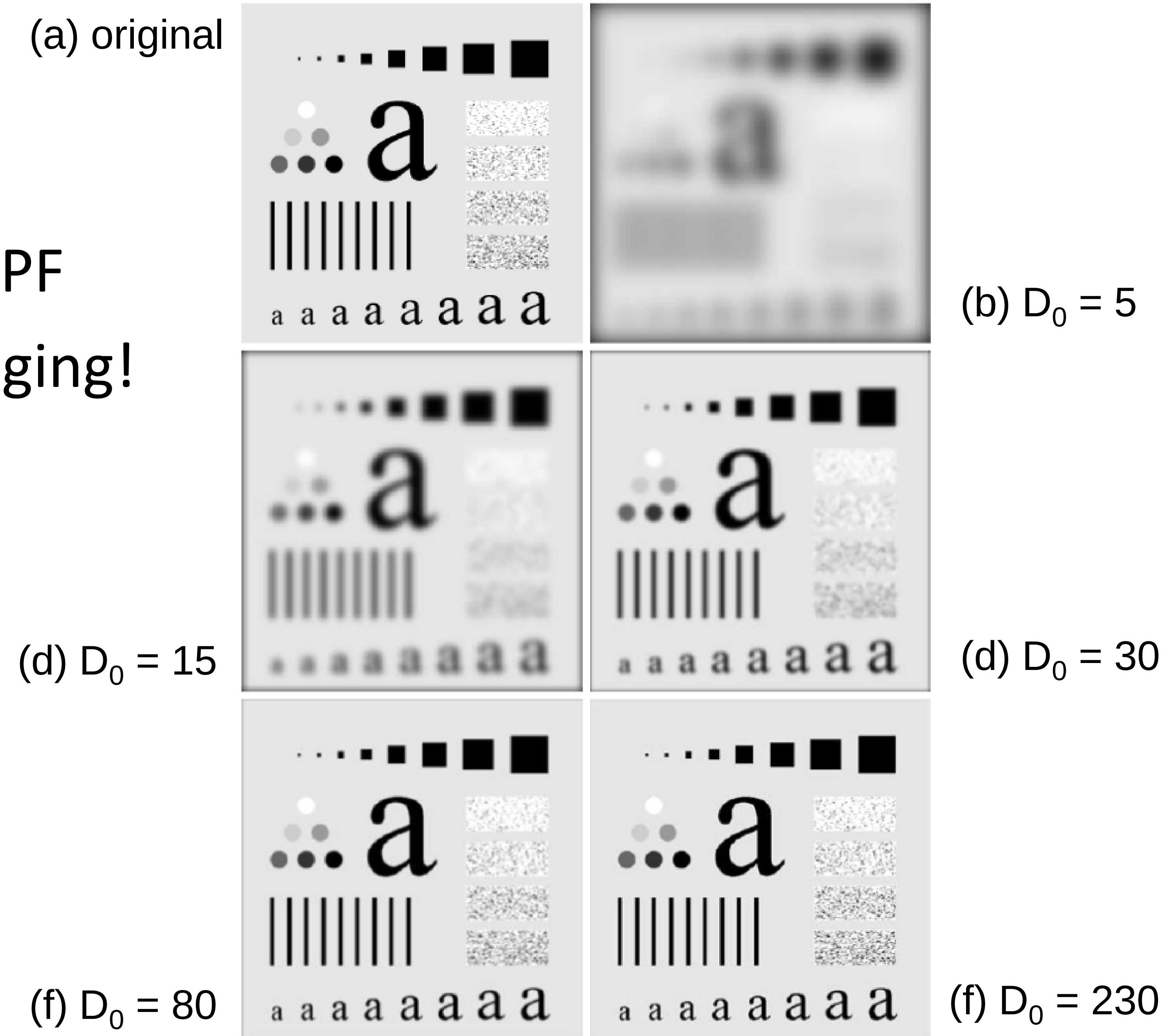




$$Gaussian \xleftrightarrow{F} Gaussian$$

Gaussian LPF

Also no ringing!



沙龍照都要這麼做…

a b c

FIGURE 4.20 (a) Original image (1028 × 732 pixels). (b) Result of filtering with a GLPF with $D_0 = 100$. (c) Result of filtering with a GLPF with $D_0 = 80$. Note reduction in skin fine lines in the magnified sections of (b) and (c).

High Pass Filters

$$H_{hp}(u,v) = 1 - H_{lp}(u,v)$$

- IHPF

$$H(u,v) = \begin{cases} 0, & if \quad D(u,v) \le D_0 \\ 1, & if \quad D(u,v) > D_0 \end{cases}$$

- Butterworth Filter

$$H(u,v) = \frac{1}{1 + \left[D_0 / D(u,v)\right]^{2n}}$$

Frequency domain

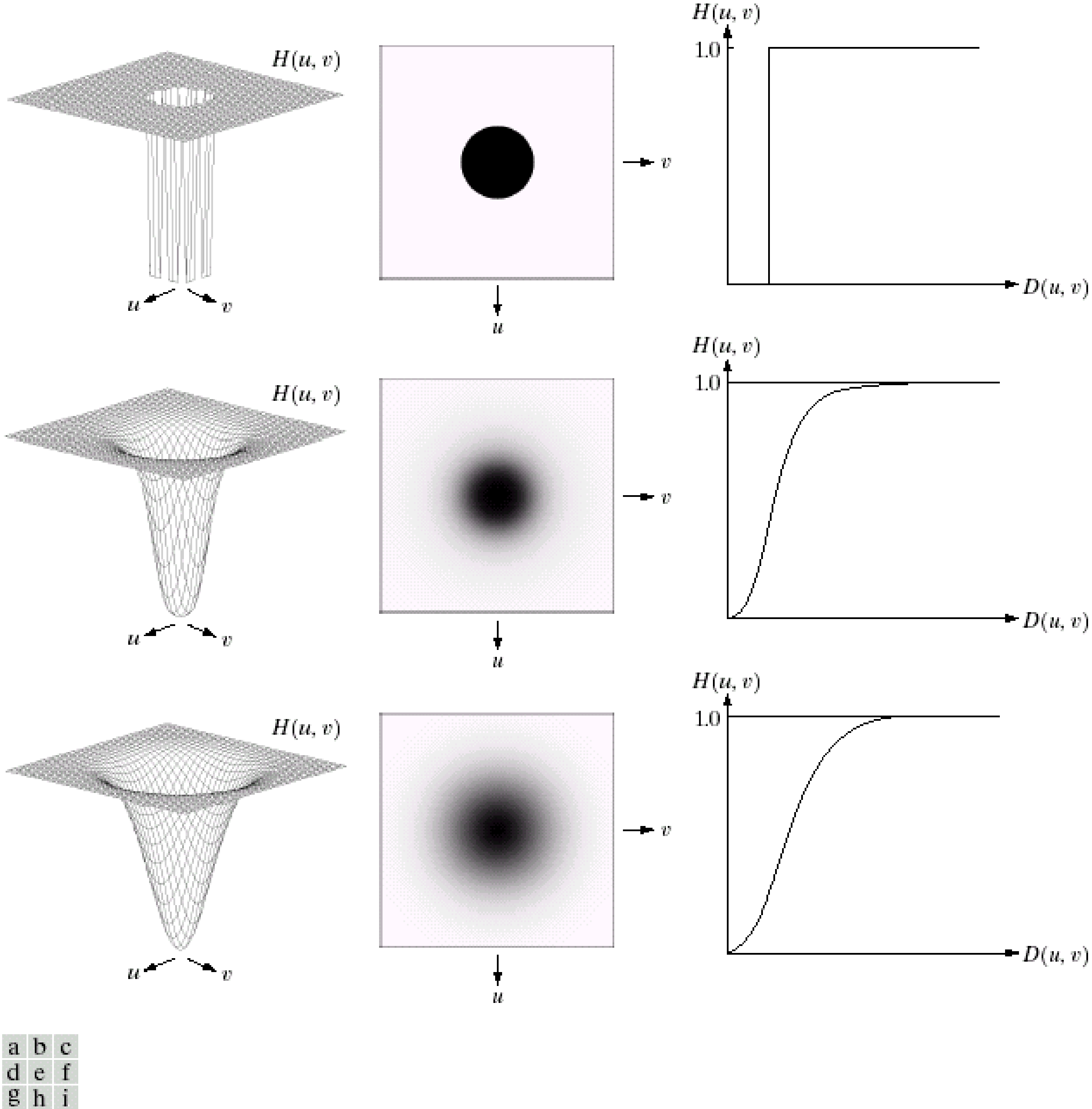


FIGURE 4.22 Top row: Perspective plot, image representation, and cross section of a typical ideal highpass filter. Middle and bottom rows: The same sequence for typical Butterworth and Gaussian highpass filters.

Image domain

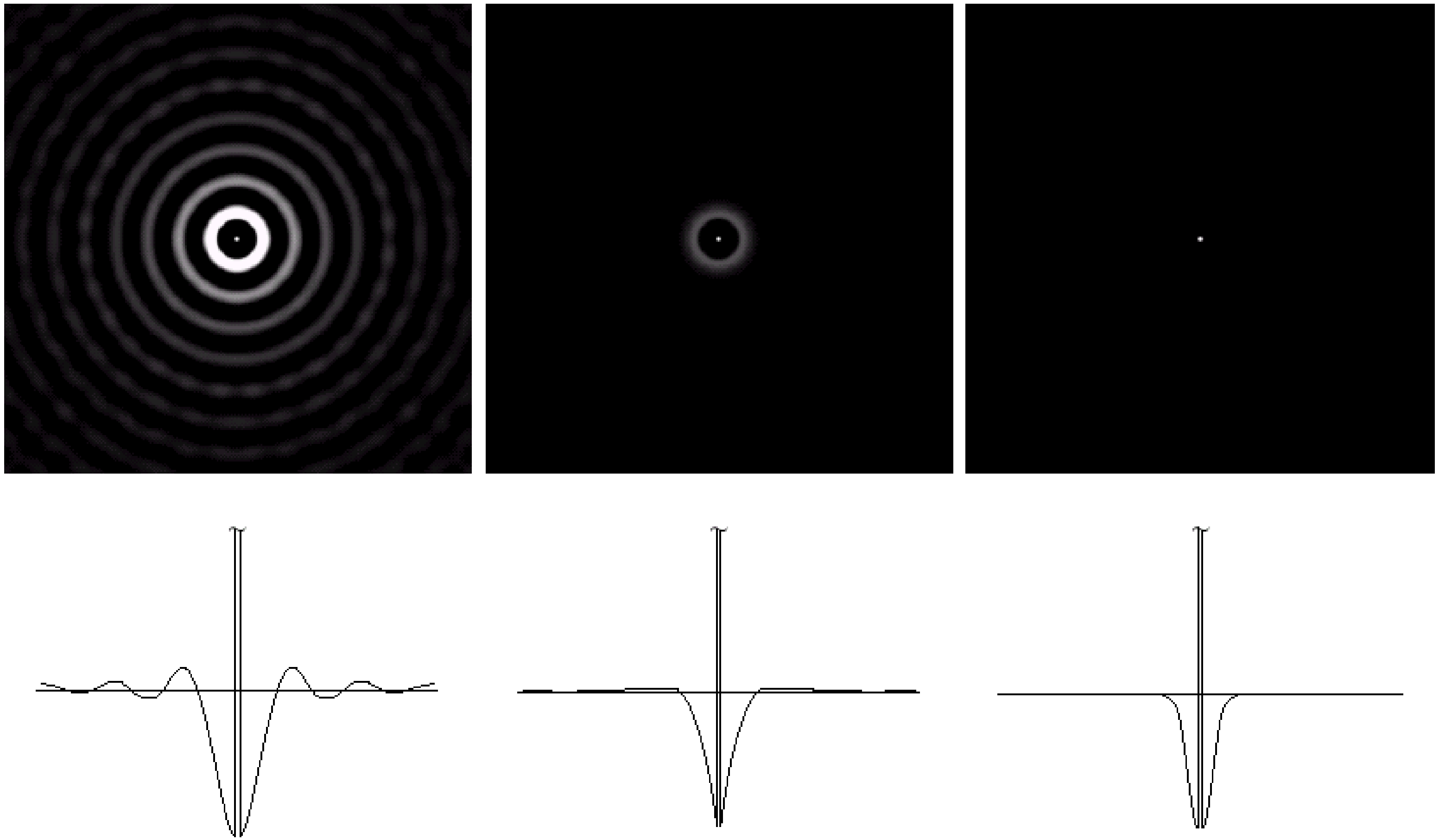
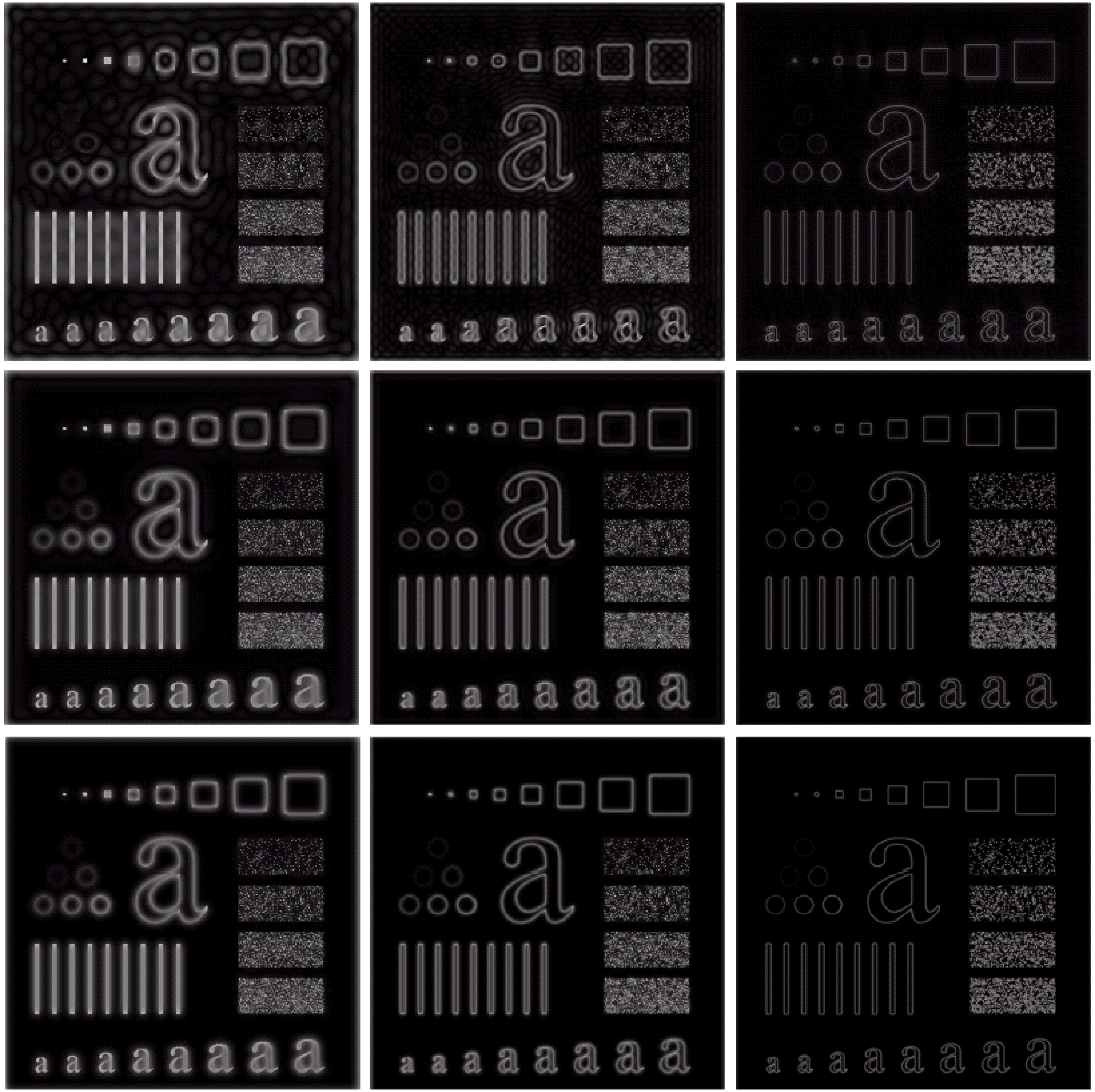


FIGURE 4.23 Spatial representations of typical (a) ideal, (b) Butterworth, and (c) Gaussian frequency domain highpass filters, and corresponding gray-level profiles.

IHPF

BHPF

GHPF



$D_0 = 30$ $D_0 = 60$ $D_0 = 160$

Application of HPF on X-ray image

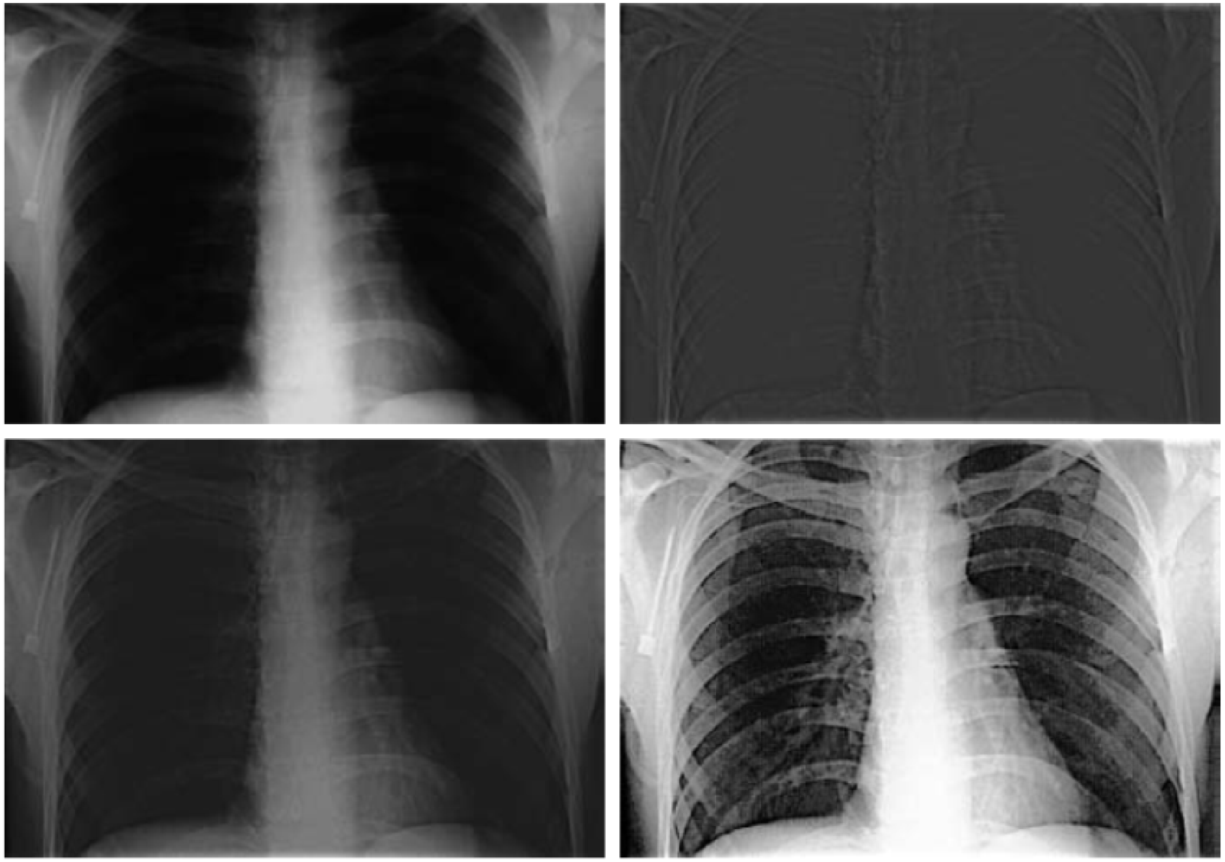


FIGURE 4.59 (a) A chest X-ray image. (b) Result of highpass filtering with a Gaussian filter. (c) Result of high-frequency-emphasis filtering using the same filter. (d) Result of performing histogram equalization on (c). (Original image courtesy of Dr. Thomas R. Gest, Division of Anatomical Sciences, University of Michigan Medical School.)

Laplacian in frequency domain

- In image domain,

$$\nabla^2 f(x,y) = \frac{\partial^2 f(x,y)}{\partial x^2} + \frac{\partial^2 f(x,y)}{\partial y^2}$$

Fourier
transform

- In frequency domain,

$$-(u^2+v^2)\cdot F(u,v) = H(u,v)\cdot F(u,v)$$

$$H(u,v) = -(u^2+v^2)$$

Selective filtering

- Bandreject and bandpass filters
  - Specific bands of frequency components are rejected or passed after filtering.

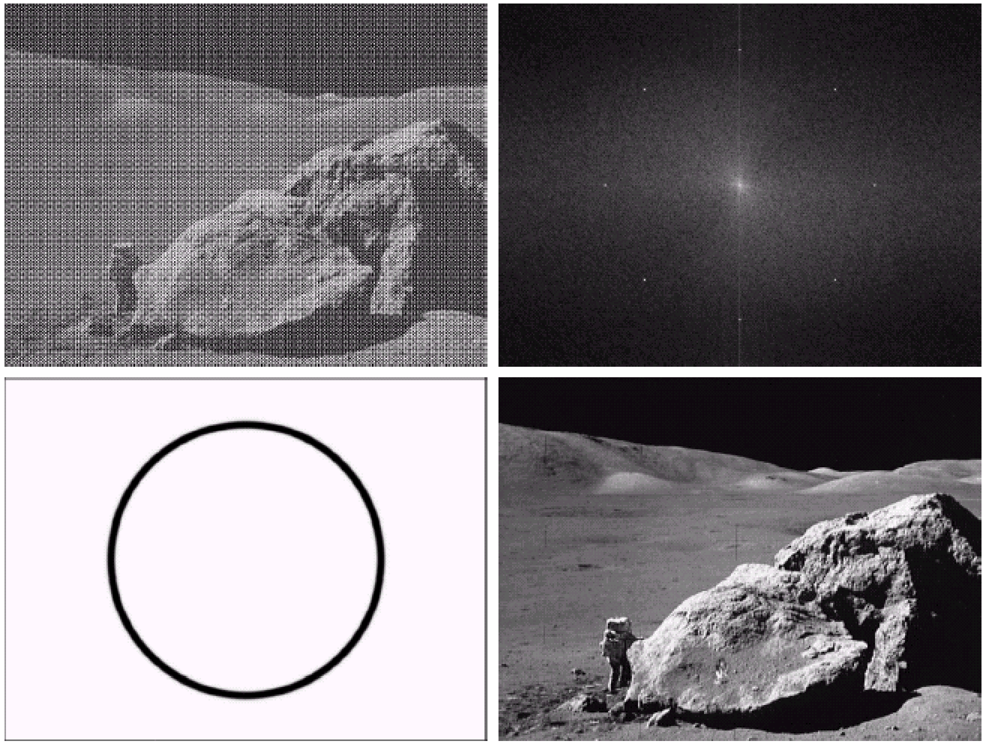
$$H_{BP}(u, v) = 1 - H_{BR}(u, v)$$

H_{BR} H_{BP}

- Notch filter
  - One or more small regions are passed or rejected.

Bandreject filters

Ideal	Butterworth	Gaussian
$H(u,v) = \begin{cases} 0 & \text{if } D_0 - \frac{W}{2} \leq D \leq D_0 + \frac{W}{2} \\ 1 & \text{otherwise} \end{cases}$	$H(u,v) = \dfrac{1}{1 + \left[\dfrac{DW}{D^2 - D_0^2}\right]^{2n}}$	$H(u,v) = 1 - e^{-\left[\frac{D^2 - D_0^2}{DW}\right]^2}$



Notch filter

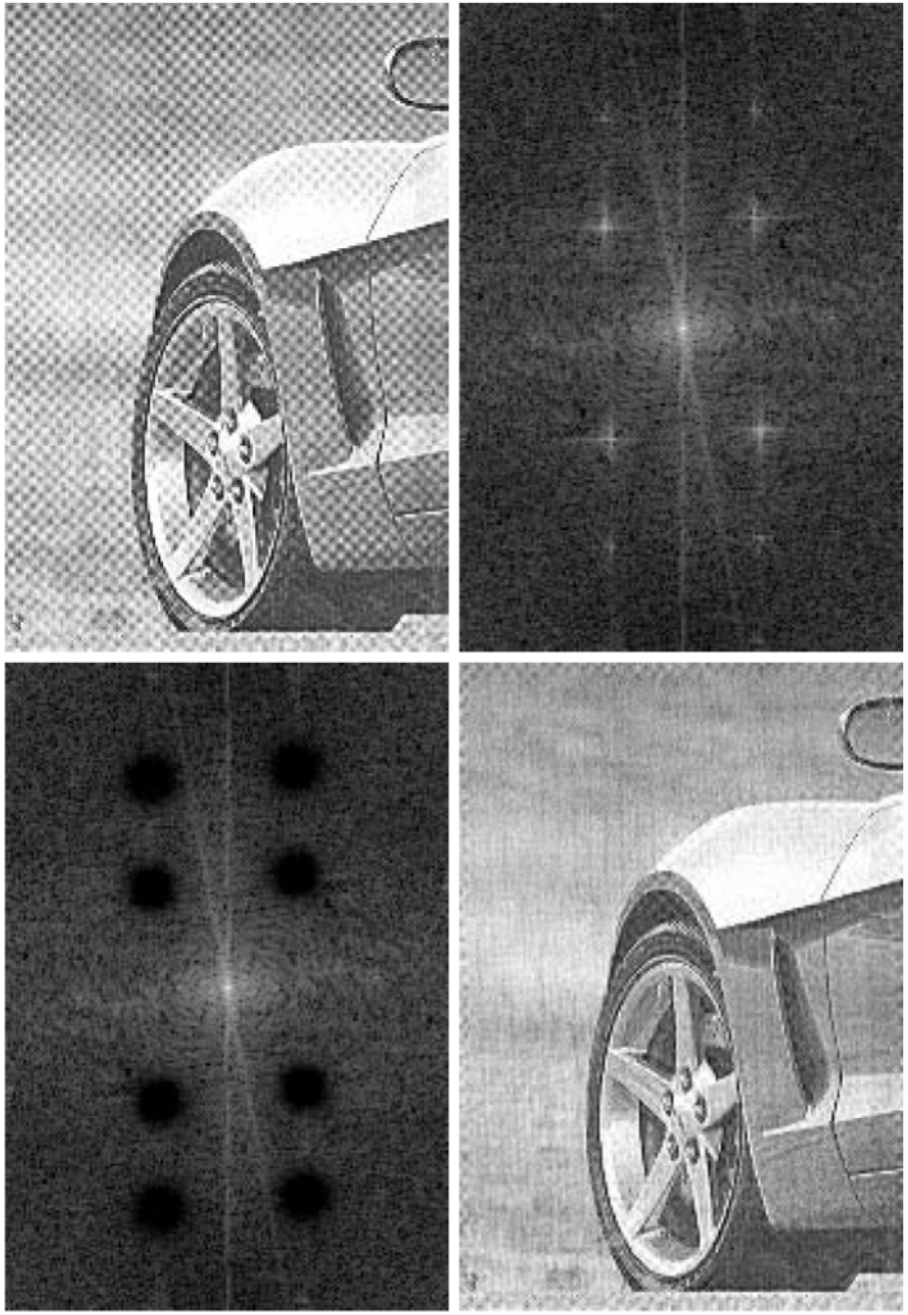
- Zero-phase-shift filter → symmetric about the center of frequency domain
- Notch reject filter:

$$H_{NR}(u,v) = \prod_{k=1}^{Q} H_k(u,v) H_{-k}(u,v)$$

  – $H_k(u, v)$ and $H_{-k}(u, v)$ are HPFs whose centers are at (u_k, v_k) and $(-u_k, -v_k)$

Example:

Notch reject filter



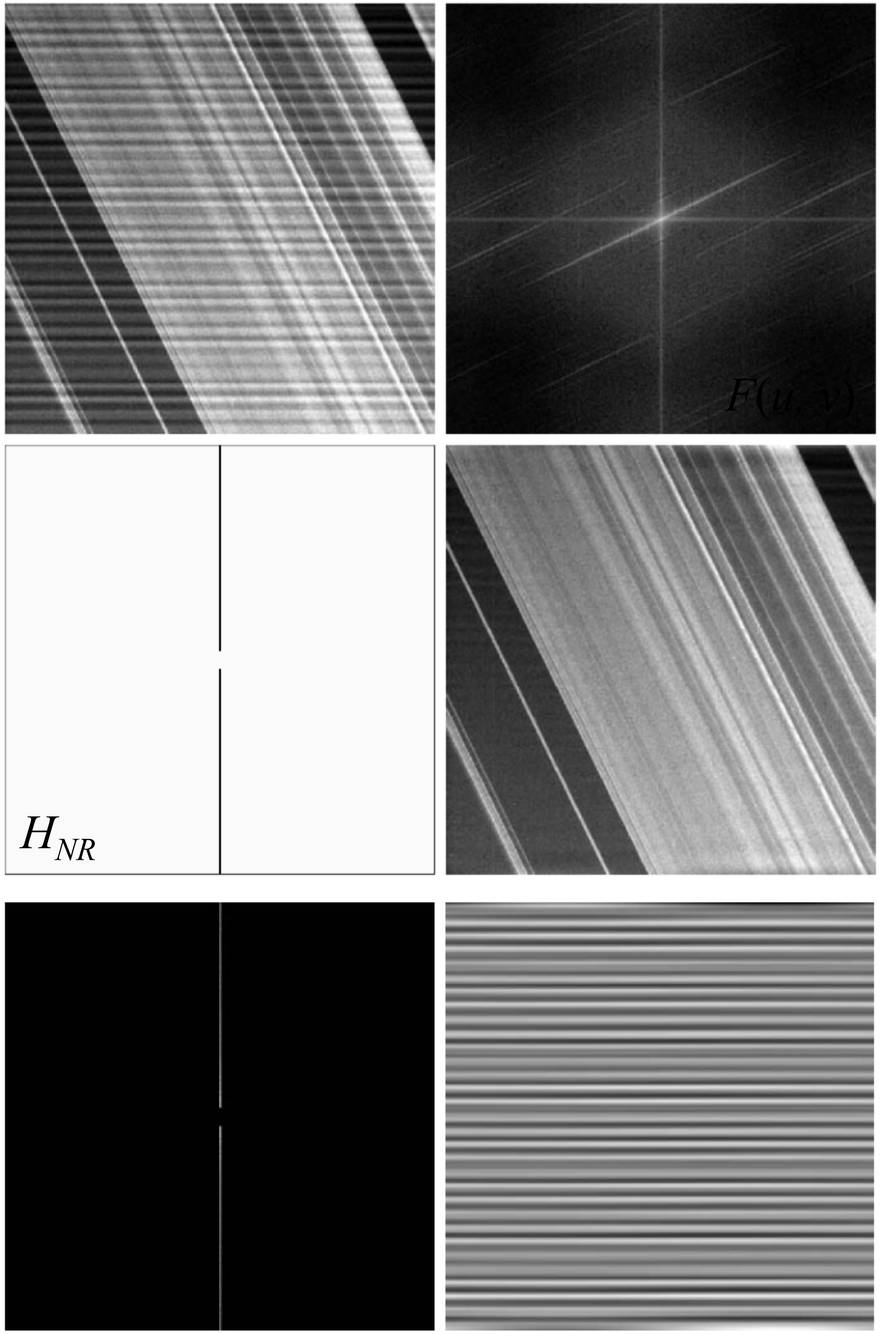
a b
c d

FIGURE 4.64
(a) Sampled newspaper image showing a moiré pattern.
(b) Spectrum.
(c) Butterworth notch reject filter multiplied by the Fourier transform.
(d) Filtered image.

$f(x, y)$:
674*674 image of
Saturn rings

Notch reject filter

Notch pass filter



Review

- Basics of signal processing
  - Convolution
  - Fourier transform
  - Sampling theorem
- Image enhancement in frequency domain
  - LPF, HPF, selective filtering
  - 絕對不只這些…

生醫影像研究方法：
空間頻譜與頻譜濾波